ON A CLASS OF MEROMORPHIC FUNCTIONS AND ITS ASSOCIATED EXCEPTIONAL SETS

BY

LENNART CARLESON

UPPSALA 1950
APPELBERGS BOKTRYCKERI AB

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Inaugural Dissertation

by

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In publishing this dissertation I take the opportunity of thanking my teachers of mathematics at the University of Uppsala. The suggestion for this thesis I owe to Professor Arne Beurling, and I should like to express my deeply felt gratitude to him for the valuable guidance he has given me in this work, as well as for the many advantages I have enjoyed in all respects as his student.

My thanks are also due to Mr. John Wermer, who has kindly undertaken to revise my English text.

Uppsala, February 1950.

Lennart Carleson.

INTRODUCTION

1. The systematic introduction of real methods into the theory of analytic functions of a complex variable constituted a very important progress in that theory. One of the most striking examples of this is the modern theory of meromorphic functions, which to a great extent is the result of the investigations of R. Nevanlinna.

Suppose that D is a simply-connected domain whose boundary consists of more than one point, and fix some point z_0 in D. If $G(z\,;z_0)$ is the Green's function for the domain D, singular at z_0, we consider the regions D_t whose boundaries are the level-curves $G(z\,;z_0)=t$. A function $w(z)$, meromorphic in D, gives a representation of D on a Riemann surface. If we consider this surface as lying on the Riemann sphere, the rate of growth of the area $S(t)$ of the image of D_t, as $t\to 0$, gives remarkably precise information about the function $w(z)$. Since $S(t)$ is invariant under any conformal mapping, it is sufficient to consider the case when D is the simple circle $|z|<1$ and we may suppose $z_0=0$. The level-curves are then the circles $|z|=r$ and it is convenient to introduce the spherical area $A(r)$ of the image of $|z|<r$, $A(r)$ being equal to $S\left(\log\frac{1}{r}\right)$. This quantity which gives us information on the general behaviour of the function has an exact relation to its properties at individual points: this is the significance of the first fundamental theorem of Nevanlinna in its invariant form given by Ahlfors and Shimizu. In fact, let $n(r,a)$ be the number of zeros of $w(z)-a$ in $|z|\leq r$ and $[a\,;b]$ the length of the chord between the points a and b on the Riemann sphere,

$$[a\,;b]=\frac{|a-b|}{\sqrt{1+|a|^2}\ \sqrt{1+|b|^2}},$$

then, if $w(z)\neq a$ on $|z|=r$,

$$\frac{d}{dr}\ \frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{[w(re^{i\Theta})\,;a]}\,d\Theta+\frac{n(r,a)}{r}=\frac{1}{\pi}\cdot\frac{A(r)}{r}, \tag{0.1}$$

$A(r)$, by definition, being independent of a. If (0.1) is integrated from 0 to r and appropriate integration constants are introduced, we get the first fundamental theorem

$$m(r, a) + N(r, a) = \frac{1}{\pi}\int_0^r \frac{A(t)}{t}\,dt \equiv T(r) \tag{0.2}$$

in the usual notations. $T(r)$ is the characteristic function of $w(z)$.

2. Hitherto the growth of the quantity $T(r)$ rather than that of $A(r)$ has been the basis of investigations of meromorphic functions. In this way a very specific class of functions has arisen, the characteristic functions of which are bounded, as $r \to 1$. We shall call this class T_0 and summarize here some of its properties which will be of importance in the following.

Let $w(z)$ belong to T_0. Then the following holds:

(A) *$w(z)$ is the quotient of two bounded functions.*

(B) $\lim_{r=1-0} w(re^{i\Theta})$ *exists for every Θ except perhaps for a certain set of measure zero.*

(C) *if $w(z) \not\equiv a$, then the set of values of Θ with $\lim_{r=1-0} w(re^{i\Theta}) = a$ is of measure zero.*

(D) *for every a the series*

$$\sum_1^\infty (1 - r_\nu(a))$$

converges, if $r_\nu(a)$, $\nu = 1, 2, \ldots$, denote the moduli of the zeros of $w(z) - a$.

(E) *if $w(z)$ maps $|z| < 1$ conformally onto the universal covering surface of a domain bounded by a closed set F, then F has positive logarithmic capacity. If, conversely, F has positive capacity, the corresponding mapping-function belongs to T_0.*

All these theorems are more or less immediate consequences of the relation (0.2). There is a detailed account in Nevanlinna [1] which also contains a comprehensive bibliography.

3. In accordance with the general ideas concerning $S(t)$, which we have mentioned earlier, it is however possible to build a more limited class than T_0 by requiring a still less rapid growth of $S(t)$. It is thus natural to make the following definitions: $w(z)$, meromorphic in D, belongs to the class $T_\alpha(D)$, $0 \leq \alpha \leq 1$, if

$$\int_0 \frac{S(t)}{t^\alpha} dt < \infty, \quad 0 \leq \alpha < 1,$$

$$S(t) = O(1), \quad \alpha = 1.$$

These definitions are the basis of this dissertation. We shall state the theorems for D being the unit circle and there is no difficulty in extending them from this to the general case. The theorems (A)—(E) will be seen to have certain correspondences in the classes T_α. The concepts of "length" and "logarithmic capacity" will throughout be replaced by Hausdorff measures and generalized capacities. Both propositions and proofs will generally become more complicated, as we cannot directly apply such a simple formula as (0.2).

As the characterizing of exceptional sets is the principal task of this investigation, we have in Chapter I collected the results concerning Hausdorff measure and capacity of sets which we will make use of.

In Chapter II we state some general theorems concerning the classes T_α and their connection with certain classes of trigonometrical series. At last we consider the example of functions in T_α furnished by the Blaschke products.

Chapter III is devoted to the analogues of the theorems (A)—(C) and some related questions. In the case $\alpha = 1$, the analogues of (B) and (C) have been proved by Beurling in an important paper [1], which is the origin of this investigation.

In Chapter IV we treat the analogue of (D) which in this case is not at all trivial. Chapter V is more incomplete on account of the peculiar difficulties which are involved in the generalization of (E). It is no doubt a question of great interest to find out the definite results here and I hope to be able to return to this subject at some later time.

---

[1] Beurling [1].

CHAPTER I

ON HAUSDORFF MEASURES AND CAPACITIES OF SETS

4. With the aid of Carathéodory's theory of outer measures, Hausdorff has introduced a class of completely additive set functions, whose null-sets have certain simple metrical properties. These functions are, however, not well suited for applications to the theory of functions, mainly because they "generally" are infinite. We shall therefore introduce another set function which is better suited for our purposes and has the same null-sets.

Let $h(r)$ denote a real function, continuous and non-decreasing for $r \geq 0$ and vanishing for $r = 0$. If furthermore $r^{-2} h(r)$ is non-increasing we call $h(r)$ a measure function. If E is an arbitrary bounded set, we can cover E by a family of circles $\{C_\nu\}_1^\infty$ with radii $\{r_\nu\}_1^\infty$ and define

$$M_h(E) = \inf \sum_1^\infty h(r_\nu)$$

for all coverings of this kind. It is easily seen that this measure [1] is zero if and only if the corresponding Hausdorff measure vanishes. $M_h(E)$ has the character of an outer measure in the sense that

$$M_h(E) = \inf_{O \supset E} M_h(O),$$

where O is an open set containing E. The inner measure $\underline{M}_h(E)$ is defined in the following way:

$$\underline{M}_h(E) = \sup_{F \subset E} M_h(F),$$

when F runs through the closed subsets of E. We have this connection between the two measures, which will be of importance later on:

[1] For the sake of brevity we use the word measure for these and other set functions, though they are not additive.

Theorem 1.1: *For any open set O it is true that*

$$\underline{M}_h(O) \leq M_h(O) \leq 32 \cdot \underline{M}_h(O),$$

where $h(r)$ is any measure function.

We introduce a sequence of nets $\mathfrak{N} = \{\mathfrak{N}_k\}$, where $\mathfrak{N}_0$ consists of all closed squares with corners having integer coordinates, and where $\mathfrak{N}_i$ consists of all the closed squares with sides of length 2^{-i} we get by dividing the squares in $\mathfrak{N}_{i-1}$ into 4 equal squares, $i > 0$. We define an auxiliary h-measure $\mathfrak{M}_h(E)$ with respect to $\mathfrak{N}$ by covering E by squares in $\mathfrak{N}$ with sides of length $2\,\delta_k$, and put

$$\mathfrak{M}_h(E) = \inf \sum h(\delta_k)$$

for all such coverings. It is easily seen that

$$\frac{1}{2} M_h(E) \leq \mathfrak{M}_h(E) \leq 16 \cdot M_h(E). \tag{1.1}$$

We can express the open set O as a union of closed squares belonging to $\mathfrak{N}$:

$$O = \sum_1^\infty \omega_\nu .$$

Then

$$F_n = \sum_1^n \omega_\nu$$

is a closed subset of O, and putting

$$\mathfrak{M}_h(F_n) = a_n ,$$

we get a non-decreasing sequence $\{a_n\}_1^\infty$ such that $a_n \leq \mathfrak{M}_h(O)$ since $F_n \subset O$. We assert

$$\lim_{n=\infty} a_n = \mathfrak{M}_h(O) ,$$

which will prove the theorem, as we then get by (1.1)

$$\lim_{n=\infty} 16\, M_h(F_n) \geq \frac{1}{2} M_h(O).$$

It is easily seen that the lower bound defining a_n is attained and there exists a family $\{R_{\nu n}\}$, $1 \leq \nu \leq N_n$, of squares in $\mathfrak{N}$ such that N_n has its least value and

$$\sum_1^{N_n} h(\delta_{\nu n}) = a_n$$

$$\sum_1^{N_n} R_{\nu n} \supset F_n.$$

If $n \geq \mu$, there is a square $R_{\nu n}$ which contains ω_μ and has a side which is at least as long as that of ω_μ. This is an immediate consequence of the hypothesis concerning $r^{-2} h(r)$. We suppose that the squares $R_{\nu n}$ for each n are numbered so that the sequence $\{\delta_{\nu n}\}$ is non-increasing.

By the diagonal process we choose the sequence of indices $\{n_i\}_1^\infty$ so that the squares $R_{\nu n_i}$ converge to the squares R_ν with sides $2\delta_\nu$ as $i \to \infty$, for all ν. Then

$$\sum_1^\infty h(\delta_\nu) \leq \lim_{n=\infty} \sum_{\nu=1}^{N_n} h(\delta_{\nu n}) = \lim_{n=\infty} a_n \leq \mathfrak{M}_h(O).$$

We assert that

$$\sum_1^\infty R_\nu \supset O,$$

which will prove the theorem. But this is evident. Let P be any point in O and suppose that P belongs to ω_μ. For each $n \geq \mu$, $R_{\nu n}$ exists including P and the length of the side of this square is not less than that of ω_μ. Hence the index ν of this square $R_{\nu n}$ is bounded independently of n because of the numeration of the squares. P must thus belong to one of the limit squares and the theorem is proved.

5. We shall introduce one more measure which has been used in lectures by professor Beurling and is essentially equivalent to $\underline{M}_h(E)$. Let Γ_h be the class of all non-negative, completely additive functions of a set such that

Γ_h: $$\mu(r, a) \leq h(r)$$

for all a, where $\mu(r, a)$ is is the value taken by μ for the circle $|z - a| \leq r$ and $h(r)$ is any fixed h-function. For a Borel set E we define

$$M_h^*(E) = \sup_{\mu \in \Gamma_h} \mu(E).$$

If E is a closed set, there exists a set function μ realizing the upper bound. This is a consequence of the selection theorem for set functions.[1] We also observe that, in taking the upper bound, we may consider only those μ in Γ_h vanishing outside E. The relation between M_h^* and $\underline{M}_h$ is contained in the following theorem.

Theorem 1.2: *If E is a bounded Borel set, then*

$$M_h^*(E) \leq \underline{M}_h(E) \leq 36\, M_h^*(E).$$

The left inequality is quite simple. Let $\mu \in \Gamma_h$. There exists a closed subset F of E such that $\mu(F) \geq (1-\varepsilon)\,\mu(E)$. If $\{C_\nu\}_1^\infty$ represents a covering of F by circles with radii $\{r_\nu\}_1^\infty$, we have

$$\sum_1^\infty h(r_\nu) \geq \sum_1^\infty \mu(C_\nu) \geq (1-\varepsilon)\,\mu(E)$$

and hence

$$M_h(F) \geq (1-\varepsilon)\,\underline{\mu}(E).$$

Taking the upper bound we obtain the desired inequality.

The right-hand inequality has been proved by Frostman[2] without stating it explicitly. We use it to give a simple proof of a theorem of H. Cartan[3] in a somewhat generalized form.

Theorem 1.3: *Let $h(t)$ and $1/g(t)$ be two functions, continuous and non-decreasing for $t \geq 0$, which vanish for $t = 0$. If $f(\Theta)$ is any measurable function, defined on a measurable set A of positive Lebesgue measure*[4]*, then for all points a such that*

$$|f(\Theta) - a| \geq \sigma > 0$$

on A, we have

$$\frac{1}{m_1 A}\int_A g(|f(\Theta)-a|)\, d\,\Theta < \int_\sigma^\infty g(t)\, d\, h(t) + g(\sigma)\, h(\sigma)$$

except perhaps when a belongs to an exceptional set E with

$$M_h(E) \leq 36.$$

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[1] See de la Vallée Poussin [1].

[2] Frostman [1] p. 87 ff.

[3] Cartan [1].

[4] We will by m_1 and m_2 denote Lebesgue measure on the line and in the plane.

E is a closed set. Let μ belong to Γ_h and vanish outside E. Then

$$\mu(E)\left\{\int_\sigma^\infty g(t)\,d\,h(t)+g(\sigma)\,h(\sigma)\right\}\leq\frac{1}{m_1\,A}\int_A d\,\Theta\int_E g(|f(\Theta)-a|)\,d\,\mu(a)=$$

$$=\frac{1}{m_1\,A}\int_A d\,\Theta\int_\sigma^\infty g(r)\,d\,\mu(r,f(\Theta)). \tag{1.2}$$

Integrating by parts we obtain

$$\int_\sigma^\infty g(r)\,d\,\mu(r,f)=g(R)\,\mu(E)-\int_\sigma^R \mu(r,f)\,d\,g(r)\leq$$

$$\leq g(R)\,\mu(E)-\int_\sigma^R h(r)\,d\,g(r)\leq$$

$$\leq\int_\sigma^\infty g(r)\,d\,h(r)+g(\sigma)\,h(\sigma).$$

If we insert this estimation in (1.2) we get

$$M_h^*(E)=\sup_{\mu\in\Gamma_h}\mu(E)\leq 1,$$

and our proposition follows from theorem 1.2.

With regard to later applications we shall introduce some special notations: for the measure function

$$h(r)=\begin{cases}\left[\log\dfrac{1}{r}\right]^{-p}, & r<e^{-p},\\ p^{-p}, & r\geq e^{-p},\end{cases}\qquad 0<p<\infty,$$

we use the notation

$$M_h(E)=l_p(E).$$

The sets E for which $l_p(E)=0$ are extremely small and such sets with $p<1$ have, as far as I know, not previously been studied in the theory of functions. If for a set E $l_{p+\varepsilon}(E)=0$ and $l_{p-\varepsilon}(E)>0$ for every $\varepsilon>0$, we say that E has the (outer) logarithmic dimension p:

$$\text{Log-dim}\,(E)=p.$$

We also set

$$M_h(E) = L_\alpha(E), \quad 0 < \alpha < 2, \quad \text{if} \quad h(r) = r^\alpha,$$

$$l_1(E) = L_0(E),$$

and denote the other measures in a similar way.

6. For the Nevanlinna-Frostman theory of the class T_0 the concept of capacity was of great importance. With regard to later applications we shall also define generalized capacities.[1]

Given a bounded Borel set E we consider the distributions μ of the unit mass on E, i.e. the completely additive, non-negative set functions, taking the value 1 on E and vanishing outside E. We introduce the potentials

$$\left.\begin{matrix} u_0^{(\mu)}(z) \\ u_\alpha^{(\mu)}(z) \end{matrix}\right\} = \int_E \left\{ \begin{matrix} \log \dfrac{1}{|z-\zeta|} \\ \dfrac{1}{|z-\zeta|^\alpha} \end{matrix} \right\} d\mu(\zeta), \quad 0 < \alpha < 2,$$

and then

$$V_\alpha = \inf_\mu \sup_z u_\alpha^{(\mu)}(z), \quad 0 \leq \alpha < 2.$$

The capacity of order α or the α-capacity is then defined as

$$C_\alpha(E) = \frac{1}{V_\alpha}.$$

$V_\alpha = \infty$ corresponds to sets with α-capacity zero and these null-sets have the property that a countable union of them has α-capacity zero. If $V_\alpha < \infty$ and E is a closed set, there exists a unique distribution μ_α such that the corresponding potential $u_\alpha(z)$ is $\leq V_\alpha$ everywhere and $= V_\alpha$ for every $z \in E$ except perhaps for a set with α-capacity zero.

We will sometimes use the term capacity and the notation $C(E)$ in the logarithmic case $\alpha = 0$, which is particularly important. The potential $u_0(z)$ with respect to $\mu_0(e)$, the Robin distribution, has the following connection with the Green's function for the complement of E, singular at infinity:

$$G(z; \infty) = V_0 - u_0(z).$$

---

[1] For the following see Frostman [1].

The measure $C_a(E)$ thus defined is an inner measure:

$$C_a(E) = \sup_{F \subset E} C_a(F).$$

Since our purpose is to show that certain sets E are small, it is important to define outer measures. The outer capacity of E is thus given by

$$\overline{C}_a(E) = \inf_{O \supset E} C_a(O).$$

The dimension of a set E, defined as above, is the same for $\overline{C}_a(E)$ and $L_a(E)$ and we denote it simply

$$\text{Dim } (E) = a.$$

This follows from the following results which show the intimate connection between the two measures.

If $\overline{C}_a(E) > 0$, $0 \leq a < 2$, then $L_a(E) > 0$.

If for an h-function $h(r)$, such that

$$\int_0 \frac{h(t)}{t^{1+a}} dt < \infty,$$

$M_h(E) > 0$, then $\overline{C}_a(E) > 0$. Analogous results hold for the inner measures.

The first of these propositions is nothing but a special case of the following theorem, which is almost trivial.

Theorem 1.4: *If $1/q(x)$ is an h-function and E is any bounded Borel set, and*

$$\int_E q(|z - \zeta|)\, d\mu(\zeta) \leq K,$$

where μ is a distribution of the unit mass on E, then

$$M_{\frac{1}{q}}(E) \geq \frac{1}{K}.$$

For any point a we have

$$q(r)\, \mu(r, a) \leq K,$$

and we conclude that $\frac{1}{K} \cdot \mu \in \Gamma_{\frac{1}{q}}$ whence

$$M^*_{\frac{1}{q}}(E) \geq \frac{1}{K},$$

and the proposition follows from theorem 1.2.

Finally we make the remark that an unbounded set E is said to have the measure zero with respect to any of our measures, if this holds for every bounded subset of E.

7. The measures thus defined have an interesting connection with harmonic and analytic functions satisfying a Lipschitz condition. A function $\varphi(z)$ is said to belong to Lip α, $0 < \alpha \leq 1$, if there exists a constant M such that

$$|\varphi(z) - \varphi(z')| \leq M |z - z'|^{\alpha} \tag{1.3}$$

for any points z and z'.

Theorem 1.5: *If F is a compact set, there exists a distribution μ of the unit mass on F such that*

$$u(z) = \int_F \log \left| \frac{1}{z - \zeta} \right| d\mu(\zeta) \in \text{Lip } \alpha, \tag{I}$$

if $L_\alpha(F) > 0$, $0 < \alpha < 1$,

$$f(z) = \int_F \frac{d\mu(\zeta)}{\zeta - z} \in \text{Lip } (\alpha - 1), \tag{II}$$

if $C_\alpha(F) > 0$, $1 < \alpha < 2$. *There is a constant M satisfying* (1.3), *which only depends on the respective measures.*

Proof (*I*): It follows from the hypothesis that there exists a distribution μ on F for which

$$\mu(r, a) \leq A r^{\alpha}.$$

We shall use the letter A in the proof to represent any quantity only depending on $L_\alpha(F)$. We can write

$$u(z) = \int_0^{\infty} \log \frac{1}{r} d\mu(r, z) = \log \frac{1}{R(z)} + \int_0^{R(z)} \frac{\mu(r, z)}{r} dr,$$

where $R(z)$ is the least number for which $\mu(R, z) = 1$. If $|z - z'| = \delta$ and $R = \min(R(z), R(z')) \geq 2\delta$, we get

$$u(z) - u(z') \leq A\,\delta + \int_0^{2\delta} + \int_{2\delta}^{R} \frac{\mu(r, z) - \mu(r, z')}{r}\, dr \leq$$

$$\leq A\,\delta + A \int_0^{2\delta} \frac{r^\alpha}{r}\, dr + \int_{2\delta}^{R} \frac{\mu(r, z) - \mu(r, z')}{r}\, dr.$$

If we observe that $\mu(r, z') \geq \mu(r - \delta, z)$, we can conclude that

$$u(z) - u(z') \leq A\,\delta^\alpha + \int_{2\delta}^{R} \frac{\mu(r, z) - \mu(r - \delta, z)}{r}\, dr \leq$$

$$\leq A\,\delta^\alpha + \int_{2\delta}^{R} \mu(r, z) \left\{ \frac{1}{r} - \frac{1}{r + \delta} \right\} dr \leq$$

$$\leq A\,\delta^\alpha + \delta \cdot \int_{2\delta}^{R} \frac{\mu(r, z)}{r^2}\, dr \leq A\,\delta^\alpha,$$

and since z and z' may be interchanged

$$|u(z) - u(z')| \leq A\,\delta^\alpha,$$

and the proposition (I) is proved.

(*II*): We choose for μ the distribution μ_α, defined above:

$$\int_F \frac{d\mu(\zeta)}{|z - \zeta|^\alpha} \leq V_\alpha = \frac{1}{C_\alpha(F)}$$

for all z. Then

$$|f(z) - f(z')| \leq \int_F \frac{|z - z'|}{|z - \zeta|\,|z' - \zeta|}\, d\mu(\zeta) \leq$$

$$\leq |z - z'|^{\alpha - 1} \int_F \frac{\{|z - \zeta| + |z' - \zeta|\}^{2-\alpha}}{|z - \zeta| \cdot |z' - \zeta|}\, d\mu(\zeta) \leq$$

$$\leq |z-z'|^{a-1}\left\{\int_F \frac{d\mu(\zeta)}{|z-\zeta|\cdot|z'-\zeta|^{a-1}}+\int_F \frac{d\mu(\zeta)}{|z'-\zeta|\cdot|z-\zeta|^{a-1}}\right\}\leq$$

$$\leq 2\,V_a\cdot|z-z'|^{a-1}$$

as is shown by an application of Hölder's inequality.

There are converses of these propositions. A separate investigation will be published elsewhere.

We end this chapter with the following simple theorem.

Theorem 1.6: *If* $F(r;\Theta)$, $0\leq r<1$, $0\leq\Theta\leq 2\pi$, *is a family of Lebesgue measurable functions of* Θ, *then*

$$\lim_{r=1-0}\int_0^{2\pi}\frac{d\Theta}{|F(r;\Theta)-a|^p}<\infty,\quad 0<p<2,$$

except perhaps when a *belongs to a set* E_p *such that* $C_p(E_p)=0$ *and a fortiori* $m_2 E_p=0$.

Suppose, on the contrary, that $C_p(E_p)>0$. Then there exists a distribution μ of the unit mass on E_p with bounded potential of order p. But

$$\int_{E_p} d\mu(a)\lim_{r=1-0}\int_0^{2\pi}\frac{d\Theta}{|F(r;\Theta)-a|^p}\leq\lim_{r=1-0}\int_0^{2\pi}d\Theta\int_{E_p}\frac{d\mu(a)}{|F(r;\Theta)-a|^p}$$

by Fatou's lemma, and this last limit is finite, contradicting the hypothesis.

CHAPTER II

§ 1. DEFINITIONS. SOME SIMPLE RESULTS

8. If $w(z)$ is meromorphic in $|z|<1$, we introduce as before the notation

$$A(r)=\iint\limits_{|z|<r}\frac{|w'(z)|^2}{(1+|w(z)|^2)^2}\,dx\,dy,\quad z=x+iy.$$

Instead of defining our classes T_α with Green's function $\log\frac{1}{r}$ as weight, we use, in order to simplify partial integrations, the function $1-r$. Thus $w(z)$ belongs to the class T_α if

T_α: $$T_\alpha(w)\equiv\int\limits_0^1\frac{A(r)}{(1-r)^\alpha}\,dr<\infty,\quad 0\le\alpha<1,$$

T_1: $$T_1(w)\equiv\lim_{r=1-0}A(r)<\infty.$$

If $n(r,a)$ denotes the number of zeros of $w(z)-a$ in $|z|\le r$ and $d\sigma_a$ the element of area in the a-plane, we get the following representation of $T_\alpha(w)$:

$$T_\alpha(w)=\int\limits_0^1\frac{dr}{(1-r)^\alpha}\int\limits_{-\infty}^{\infty}\frac{n(r,a)}{(1+|a|^2)^2}\,d\sigma_a,\quad 0\le\alpha<1,\tag{2.1}$$

and analogously for $\alpha=1$. In order to describe the growth of area of the Riemann surface over individual points we make the following definitions. Given any point a_0 on the Riemann sphere, let s_0 be a calotte[1] with a_0 as centre. If $|s_0|$ is its area and S_0 its projection on the a-plane, we define[2]

$$q_\alpha(a_0)=\overline{\lim_{|s_0|\to 0}}\frac{1}{|s_0|}\iint\limits_{S_0}\frac{d\sigma_a}{(1+|a|^2)^2}\int\limits_0^1\frac{n(r,a)}{(1-r)^\alpha}\,dr,\quad 0<\alpha<1,$$

$$q_1(a_0)=\overline{\lim_{|s_0|\to 0}}\frac{1}{|s_0|}\iint\limits_{S_0}\frac{n(1,a)}{(1+|a|^2)^2}\,d\sigma_a.$$

---

[1] We call a domain on a sphere bounded by a circle a *calotte*.

[2] See Beurling [1] in the case $\alpha=1$.

We call a_0 α-regular if $q_\alpha(a_0)$ is finite and α-singular if it is infinite. The set of all α-singular points will in the sequel be denoted by Q_α. If $w(z) \in T_\alpha$, then Q_α has according to a well-known theorem of Lebesgue the two dimensional Lebesgue measure zero.

9. We now give some simple results concerning the classes T_α.

If $w(z) \in T_\alpha$, $0 \leq \alpha \leq 1$, *and* $R(w)$ *is a rational function, then* $w_1(z) = R(w(z)) \in T_\alpha$.

Since, $0 \leq \alpha < 1$,

$$T_\alpha(w_1) = \int_0^1 \frac{dr}{(1-r)^\alpha} \iint_{|z|<r} \frac{|R'(w(z))|^2}{(1+|R(w(z))|^2)^2} \cdot |w'(z)|^2 \, d\sigma_z,$$

we need only observe that there is a constant M so that

$$\frac{|R'(w)|}{1+|R(w)|^2} \leq \frac{M}{1+|w|^2}.$$

The same argument works in the case $\alpha = 1$.

If $w_1(z)$ *and* $w_2(z)$ *are two bounded, holomorphic functions in* $|z|<1$, $|w_i(z)| \leq 1$, *then*

$$\text{(2.2)} \qquad \begin{aligned} &\text{(a)} \quad T_\alpha(w_1+w_2) \leq 8\,T_\alpha(w_1) + 8\,T_\alpha(w_2) \\ &\text{(b)} \quad T_\alpha(w_1 \cdot w_2) \leq 8\,T_\alpha(w_1) + 8\,T_\alpha(w_2). \end{aligned}$$

We prove e. g. (b):

$$\{T_\alpha(w_1 \cdot w_2)\}^{\frac{1}{2}} \leq \left\{ \int_0^1 \frac{dr}{(1-r)^\alpha} \iint_{|z|<r} |w_1' \cdot w_2 + w_1 \cdot w_2'|^2 \, d\sigma_z \right\}^{\frac{1}{2}} \leq$$

$$\leq \sum_{i=1}^{2} \left\{ \int_0^1 \frac{dr}{(1-r)^\alpha} \iint_{|z|<r} |w_i'|^2 \, d\sigma_z \right\}^{\frac{1}{2}} \leq 2 \sum_{i=1}^{2} \{T_\alpha(w_i)\}^{\frac{1}{2}}$$

by Minkowski's inequality, and the proposition follows.

If $|w(z)| \leq 1$ *and*

$$w(z) = \sum_0^\infty a_n z^n, \quad |z|<1,$$

then

$$\text{(2.3)} \qquad m\,T_\alpha(w) \leq \sum_1^\infty |a_n|^2 n^\alpha \leq M\,T_\alpha(w),$$

where M *and* m *only depend on* α.

We have, if $k(w)$ denotes a certain constant, $\frac{1}{4} \leq k(w) \leq 1$, and $0 \leq \alpha < 1$,

$$T_\alpha(w) = k(w) \int_0^1 \frac{dr}{(1-r)^\alpha} \int_0^r r\, dr \int_0^{2\pi} |w'(r e^{i\Theta})|^2\, d\Theta =$$

$$= 2\pi k \int_0^1 \frac{dr}{(1-r)^\alpha} \int_0^r \sum_1^\infty |a_n|^2 n^2 r^{2n-1}\, dr =$$

$$= \pi k \sum_1^\infty |a_n|^2 n^\alpha \cdot n^{1-\alpha} \int_0^1 \frac{r^{2n}}{(1-r)^\alpha}\, dr,$$

and the proposition is proved, since a simple calculation shows that

$$\lim_{n=\infty} n^{1-\alpha} \int_0^1 \frac{r^{2n}}{(1-r)^\alpha}\, dr = \frac{\Gamma(1-\alpha)}{2^{1-\alpha}}.$$

The case $\alpha = 1$ is treated similarly. The left-hand inequality evidently holds without the assumption that $w(z)$ is bounded.

This result shows the intimate connection existing between our classes T_α and the classes of trigonometrical series which have been investigated by Beurling, Salem-Zygmund and Broman[1].

The following result will be of great importance for us:

A trigonometrical series $\sum_1^\infty (a_n \cos n x + b_n \sin n x)$ *converges for every* x *not belonging to a certain set of outer* $(1-\alpha)$*-capacity zero, if*

$$\sum_1^\infty (a_n^2 + b_n^2)\, n^\alpha < \infty, \quad 0 < \alpha \leq 1. \tag{2.4}$$

The convergence of the series (2.4) is equivalent to a certain continuity property of the function represented by the trigonometrical series and this fact will be used in a slightly more general form.

If $\varphi(x)$ *is a real or complex function defined by*

$$\varphi(x) \sim \frac{a_0}{2} + \sum_1^\infty (a_n \cos n x + b_n \sin n x),$$

[1] Beurling [1], Salem-Zygmund [1], Broman [1].

then for $0 < \alpha \leq 1$

$$(2.5) \qquad S_{\alpha,\beta}(\varphi) \equiv \int_0^{\frac{1}{2}} \frac{dt}{t^{1+\alpha}\left(\log \frac{1}{t}\right)^{\beta}} \int_0^{2\pi} |\varphi(x+t) - \varphi(x-t)|^2 \, dx =$$

$$= \sum_1^{\infty} (|a_n|^2 + |b_n|^2) A_n,$$

where the numbers A_n *are positive and independent of* φ *and such that . for a finite and positive* A,

$$\lim_{n=\infty} \frac{(\log n)^{\beta}}{n^{\alpha}} \cdot A_n = A.$$

Parseval's relation yields

$$S_{\alpha,\beta}(\varphi) = 4\pi \sum_1^{\infty} (|a_n|^2 + |b_n|^2) \int_0^{\frac{1}{2}} \frac{\sin^2 n t}{t^{1+\alpha}\left(\log \frac{1}{t}\right)^{\beta}} dt.$$

The integrals B_n are positive and independent of φ. Furthermore

$$\lim_{n=\infty} \frac{(\log n)^{\beta}}{n^{\alpha}} B_n = \lim_{n=\infty} \int_0^{n/2} \frac{\sin^2 \xi}{\xi^{1+\alpha}} \left(\frac{\log n}{\log \frac{n}{\xi}}\right)^{\beta} d\xi = \int_0^{\infty} \frac{\sin^2 \xi}{\xi^{1+\alpha}} \, d\xi,$$

and the proposition follows.

§ 2. ON BLASCHKE PRODUCTS

10. It is known that any bounded, holomorphic function in $|z| < 1$ can be represented as a product of one non-vanishing function and one containing all its zeros — a Blaschke product:

$$(2.6) \qquad B(z) = z^m \prod_1^{\infty} \frac{a_\nu - z}{1 - z\bar{a}_\nu} \cdot \frac{\bar{a}_\nu}{|a_\nu|}, \quad |z| < 1,$$

where $\{a_\nu\}_1^{\infty}$, $0 < |a_\nu| < 1$, is a sequence of complex numbers such that

$$(2.7) \qquad \sum_1^{\infty} (1 - |a_\nu|) < \infty.$$

In this way the Blaschke products have come to play an important role in the theory of functions, and it is therefore of great interest to consider their connection with our classes T_α.

We define a subclass B_α of the Blaschke products by requiring a certain rapidity of convergence for the series (2.7)

$$B_\alpha: \qquad \sum_1^\infty (1-|a_\nu|)^{1-\alpha} < \infty, \quad 0 \leq \alpha < 1.$$

This class has been studied by Frostman[1]. We shall for the sake of simplicity suppose that $B(0) > 0$.

Theorem 2.1: *A necessary and sufficient condition that the function $w(z)$ which is holomorphic in $|z|<1$, should belong to B_α, $0<\alpha<1$, is that*

$$\int_0^1 \int_0^{2\pi} \frac{|\log|w(re^{i\Theta})||}{(1-r)^{1+\alpha}}\, d\Theta\, dr < \infty. \tag{2.8}$$

The corresponding condition for $\alpha = 0$ is given by M. Riesz:

$$\lim_{r=1-0} \int_0^{2\pi} |\log|w(re^{i\Theta})||\, d\Theta = 0.^2$$

If $B(z)$ is a Blaschke product and $n(r)$ denotes the number of zeros in $|z| \leq r < 1$, we have, according to Jensen's theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|B(re^{i\Theta})|}\, d\Theta = \log\frac{1}{|B(0)|} - \sum_{|a_\nu|<r} \log\frac{r}{|a_\nu|} = \int_r^1 \frac{n(t)}{t}\, dt. \tag{2.9}$$

Furthermore

$$\sum_1^\infty (1-|a_\nu|)^{1-\alpha} = \alpha(1-\alpha)\int_0^1 \frac{dr}{(1-r)^{1+\alpha}} \int_r^1 n(t)\, dt + \tag{2.10}$$

$$+ (1-\alpha)\int_0^1 n(t)\, dt,$$

whether the series converges or diverges.

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[1] Frostman [2]. [2] Frostman [1].

If, thus, $w(z) \in B_\alpha$, it is an immediate consequence of (2.10) and (2.9) that the integral in (2.8) converges.

If, on the other hand, (2.8) holds, it follows from the fact that $\lim_{r=1-0} \int_0^{2\pi} |\log|w(re^{i\Theta})|| \, d\Theta$ always exists that this limit must vanish. Hence, by the theorem of M. Riesz, $w(z)$ has to be a Blaschke product and we conclude from (2.9) and (2.10) that $w(z) \in B_\alpha$, and the theorem is proved.

11. We are now going to prove that every function in B_α also belongs to T_α. We shall in fact prove something more, viz.

Theorem 2.2: *If* $w(z) \in T_\alpha$ *and* $B(z) \in B_\alpha$, *then*

$$f(z) = \frac{w(z)}{B(z)}$$

belongs to T_α.

By taking $w(z) \equiv 1$, we obtain our initial proposition. The more general theorem has a certain interest, as the quotient of two functions in T_α not necessarily belongs to this class. We construct an example in nr. 21, page 55.

For the proof we introduce the following notations:

$A(r)$ the spherical area of the image under $f(z)$ of $|z| < r$,
$\Omega(r)$ „ „ „ „ „ „ „ $w(z)$ „ $|z| < r$,
$p(r)$ the number of poles of $w(z)$ in $|z| \leq r$,
$n(r)$ „ „ „ zeros „ $B(z)$ „ $|z| \leq r$.

The relation (0.1) with $\alpha = \infty$ gives

$$\frac{1}{\pi} \cdot \frac{A(r)}{r} \leq \frac{p(r)}{r} + \frac{n(r)}{r} + \frac{1}{4\pi} \frac{d}{dr} \int_0^{2\pi} \log\left(1 + |f(re^{i\Theta})|^2\right) d\Theta,$$

where the sign of inequality holds if $w(z)$ and $B(z)$ have common zeros. By another use of (0.1) we get

$$(2.11) \quad \frac{1}{\pi} \cdot \int_0^1 \frac{A(r)}{r(1-r)^\alpha} \, dr \leq \frac{1}{\pi} \cdot \int_0^1 \frac{\Omega(r)}{r(1-r)^\alpha} \, dr + \int_0^1 \frac{n(r)\, dr}{r(1-r)^\alpha} +$$

$$+ \frac{1}{4\pi} \int_0^1 \frac{dr}{(1-r)^\alpha} \frac{d}{dr} \int_0^{2\pi} \log \frac{|B|^2 + |w|^2}{|B|^2 (1 + |w|^2)} \, d\Theta.$$

The first two terms on the right-hand side are finite according to the assumptions on $w(z)$ and $B(z)$. To find an estimation of the third term, we observe that

$$1 \leq \frac{|B|^2+|w|^2}{|B|^2(1+|w|^2)} \leq \frac{1}{|B|^2}.$$

If we now integrate by parts and use the above together with (2.9), we find that the third term is dominated by

$$\lim_{r=1-0}(1-r)^{-\alpha}\int_r^1 \frac{n(t)}{t}\,dt + \alpha\int_0^1 \frac{dr}{(1-r)^{1+\alpha}}\int_r^1 \frac{n(t)}{t}\,dt < \infty$$

because of (2.10) and our assumption. From (2.11) we obtain at last

$$\int_0^1 \frac{A(r)}{(1-r)^{\alpha}}\,dr < \infty,$$

which was our assertion.

12. Frostman has proved that if

$$\sum_{\nu=1}^{\infty} \frac{1-|a_\nu|}{|e^{i\Theta}-a_\nu|} < \infty,$$

then the Blaschke product (2.6) tends to a limit with modulus 1 as z tends to $e^{i\Theta}$ along the radius. It is an immediate consequence that if $B(z) \in B_\alpha$, this holds true for all Θ except perhaps for a set of outer $(1-\alpha)$-capacity zero.[1] It is, however, possible to prove the following sharper theorem.

Theorem 2.3: *If $B(z) \in B_\alpha$, then $B(z)$ has radial limits with modulus 1 except perhaps on an exceptional set E, such that $L_{1-\alpha}(E) = 0$.*

On account of the connection between Hausdorff measures and capacities this theorem contains Frostman's.

The proposition is a very special case of the following general theorem concerning Borel series.

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[1] Frostman [2].

Theorem 2.4: *Let* $\{z_\nu\}_1^\infty$ *be a sequence of complex numbers in the unit circle and* $\{A_\nu\}_1^\infty$ *a sequence of positive numbers less than* 1. *If* $h(r)$ *is a measure function and*

$$\sum_{\nu=1}^{\infty} A_\nu \int_{A_\nu}^{2} \frac{h(r)}{r^2}\, dr < \infty, \tag{2.12}$$

then the series

$$\sum_{\nu=1}^{\infty} \frac{A_\nu}{|z - z_\nu|}$$

converges when z *does not belong to an exceptional set* E *with* $M_h(E) = 0$.

The proof is based on a method due to Beurling. We observe that the convergence of (2.12) implies

$$\sum_{1}^{\infty} h(A_\nu) < \infty.$$

For any integer p we denote by O_p the open set where

$$\sum_{p}^{\infty} \frac{A_\nu}{|z - z_\nu|} > 1.$$

Let μ be any set function belonging to Γ_h, i.e.

$$\mu(r, a) \leq h(r),$$

and vanishing outside O_p. Around each point z_ν, $\nu \geq p$, we put a circle with radius A_ν and call the exterior of these circles G_p. On the circles μ distributes a mass which is not greater than

$$\sum_{p}^{\infty} h(A_\nu) = \varepsilon_p'.$$

We find, $p > p_0$,

$$\mu(G_p) \leq \int_{G_p} d\mu(z) \sum_{p}^{\infty} \frac{A_\nu}{|z - z_\nu|} \leq \sum_{p}^{\infty} A_\nu \int_{A_\nu}^{3} \frac{d\mu(r, z_\nu)}{r} \leq$$

$$\leq \frac{h(3)}{3} \sum_{p}^{\infty} A_\nu + \sum_{p}^{\infty} A_\nu \int_{A_\nu}^{3} \frac{h(r)}{r^2}\, dr = \varepsilon_p''$$

and thus, $\varepsilon_p = \varepsilon'_p + \varepsilon''_p$,

$$\sup_{\mu \in \Gamma_h} \mu(O_p) = M_h^*(O_p) \leq \varepsilon_p, \ \varepsilon_p \to 0, \ p \to \infty,$$

and according to theorems 1.1 and 1.2

$$M_h(O_p) \leq 32 \underline{M}_h(O_p) \leq 32 \cdot 36 \cdot \varepsilon_p.$$

Apart from the points z_ν, $\nu = 1, 2, \ldots, p-1$, O_p contains E and we conclude

$$M_h(E) \leq \lim_{p=\infty} M_h(O_p) = 0,$$

which was our assertion.

We shall now show that this theorem, if $h(r) = r^\alpha$, $0 < \alpha < 1$, cannot be improved. Also theorem 2.3 can in an analogous way be shown to be the best possible.

Theorem 2.5: *Given a bounded set E, a necessary and sufficient condition for the existence of a Borel series*

$$\sum_{\nu=1}^{\infty} \frac{A_\nu}{|z - z_\nu|} \quad \textit{with} \quad \sum_{1}^{\infty} A_\nu^\alpha < \infty, \ 0 < \alpha < 1,$$

which diverges on E, is that

$$L_\alpha(E) = 0.$$

The necessity is exactly theorem 2.4 in this case. Suppose now that $L_\alpha(E) = 0$. For $n = 1, 2, \ldots$ we cover E by a family of circles with radii $r_{n\mu}$ so that

$$\sum_{\mu=1}^{\infty} r_{n\mu}^\alpha \leq 2^{-n}.$$

If these circles have the centres $z_{n\mu}$, the Borel series

$$\sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} \frac{r_{n\mu}}{|z - z_{n\mu}|} = \sum_{\nu=1}^{\infty} \frac{A_\nu}{|z - z_\nu|}$$

diverges in every point in E while

$$\sum_{\nu=1}^{\infty} A_\nu^\alpha = \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} r_{n\mu}^\alpha \leq \sum_{n=1}^{\infty} 2^{-n} = 1.$$

13. The Blaschke products are included in a bigger class of function $\mathfrak{U}$, introduced by Frostman.[1] A function $f(z)$, holomorphic in $|z|<1$, belongs to $\mathfrak{U}$ if

$$\mathfrak{U}: \quad \begin{cases} |f(z)|<1, \ |z|<1, \\ \lim\limits_{r=1-0} |f(re^{i\Theta})| = 1 \text{ a.e.} \end{cases}$$

It is possible to develop a general theory for the intersection $\mathfrak{U}_\alpha$ of the classes $\mathfrak{U}$ and T_α, $0<\alpha<1$. For example theorem 2.3 holds for functions in $\mathfrak{U}_\alpha$. However, we are content with an analogue of the following theorem of Frostman[1]: The set E of values in the unit circle omitted by a function $f(z) \in \mathfrak{U}$ is of logarithmic capacity zero.

Theorem 2.6: *If* $f(z) \in \mathfrak{U}_\alpha$, *then* $l_{1-\alpha}(E) = 0$.

The proposition is an immediate consequence of theorem 5.6 on page 67. We shall give another theorem concerned with $\mathfrak{U}_\alpha$ in nr. 22.

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[1] Frostman [1], page 103 ff.

CHAPTER III

§ 1. AN AUXILIARY THEOREM

14. The results of this chapter will largely depend on the following theorem.

Theorem 3.1: *Let* $u(z)$,

$$u(z) \sim \frac{a_0(r)}{2} + \sum_1^\infty (a_n(r)\cos n\,\Theta + b_n(r)\sin n\,\Theta), \quad z = r\,e^{i\Theta},$$

be a real function, continuous at $z=0$, *having continuous partial derivatives in* $|z|\leq 1$ *except on a finite number of analytic curves. Suppose furthermore that its Fourier coefficients are continuous functions of* r *for* $0\leq r\leq 1$. *Defining*

$$D_\alpha(u) \equiv \frac{1}{\pi}\int_0^1\int_0^{2\pi}\left\{\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \Theta}\right)^2\right\}(1-r)^{1-\alpha}\,d\Theta\,dr,$$

we have, $0<\alpha\leq 1$,

$$\sum_1^\infty [a_n(1)^2 + b_n(1)^2]\,n^\alpha \leq M\cdot D_\alpha(u),$$

M *depending only on* α.

We need the following

Lemma: *Suppose that* $y(x)$ *is an absolutely continuous real function for* $0\leq x\leq 1$ *and suppose that* $y(0)=0$ *and* $y(1)=1$. *Then for every* β, $0\leq\beta<1$, *there exists a constant* $m(\beta)>0$ *such that*

$$I_n(y) \equiv \int_0^1 [n^2\,y(x)^2 + y'(x)^2]\,(1-x)^\beta\,dx \geq m\cdot n^{1-\beta} \tag{3.1}$$

for all functions $y(x)$ *of this kind and all* $n\geq 1$.

We make a simple transformation of (3.1) by putting

$$t = (1-x)^{1+\beta}, \quad \gamma = \frac{2\beta}{1+\beta}, \quad k = \frac{n}{1+\beta}.$$

We get a function $z(t)$: $z(0)=1$ and $z(1)=0$, and an equivalent proposition for

$$(1+\beta)^{-1} I_n(y) = J_k(z) \equiv \int_0^1 [k^2 z(t)^2 + z'(t)^2 t^\gamma]\, dt.$$

Suppose that $\mu < \frac{1}{16}$ is a positive number such that a function $z(t)$ exists with

$$J_k(z) \leq \mu \cdot k^{1-\beta}.$$

Then $z(t) < \frac{1}{2}$ except when t belongs to a set E:

$$\frac{k^2}{4} m_1 E < \mu k^{1-\beta}$$

$$m_1 E < 4\mu k^{-(1+\beta)} = \eta < 1. \tag{3.2}$$

It is easily seen that

$$\int_E z'(t)^2 t^\gamma\, dt \geq \inf_f \int_0^\eta f'(t)^2 t^\gamma\, dt,$$

where $f(0)=1$ and $f(\eta)=\frac{1}{2}$. Solving this simple variational problem we obtain for the solution $f_0(t)$:

$$f_0'(t) = -\frac{1-\gamma}{2} \eta^{\gamma-1} \cdot t^{-\gamma},$$

and denoting by $A(\beta)$ numbers only depending on β[1] we get

$$\mu k^{1-\beta} \geq J_k(z) > \int_0^\eta f_0'(t)^2 t^\gamma\, dt = A(\beta) \eta^{-(2-2\gamma)} \int_0^\eta t^{-\gamma}\, dt = A(\beta) \cdot \eta^{-(1-\gamma)}.$$

If the value of η in (3.2) is inserted, we find

$$\mu \geq A(\beta)\, \mu^{-(1-\gamma)} \cdot k^q,$$

[1] We will in the sequel use the symbol $A(\ )$ for numbers having the character of constants in the proofs.

where

$$q = (1-\gamma)(1+\beta) - (1-\beta) = \frac{(1-\beta)(1+\beta)}{1+\beta} - (1-\beta) = 0$$

and

$$\mu \geq A(\beta).$$

The lemma is thus proved.

To prove the theorem, we observe that $D_\alpha(u)$ does not increase if we remove the factor $\frac{1}{r^2}$. By Parseval's relation and our assumptions we get

$$D_\alpha(u) \geq \int_0^1 \left\{ \frac{a_0'(r)^2}{2} + \sum_1^\infty [a_n'(r)^2 + n^2 a_n(r)^2] \right\} (1-r)^{1-\alpha}\, dr +$$

$$+ \textit{ the corresponding terms for the sine-coefficients} \geq$$

$$\geq \sum_1^\infty \int_0^1 [n^2 a_n(r)^2 + a_n'(r)^2] (1-r)^{1-\alpha}\, dr + \textit{corresp. terms}.$$

If we use the lemma on each term in the sums and observe that

$$a_n(0) = b_n(0) = 0, \quad n \geq 1,$$

as $u(z)$ is continuous at $z=0$, we finally get

$$D_\alpha(u) \geq m(1-\alpha) \cdot \sum_1^\infty [a_n(1)^2 + b_n(1)^2]\, n^\alpha,$$

which was our assertion.

§ 2. ON RADIAL LIMITS

15. The theorem cited in nr. 9 concerning convergence of certain classes of trigonometrical series combined with (2.3) gives us the result that any bounded function $w(z)$, belonging to T_α, has radial limits except perhaps on a set whose outer $(1-\alpha)$-capacity is zero. We are going to prove that this holds true for any function in T_α. In the case $\alpha = 1$ this has been shown by Beurling.[1]

[1] Beurling [1].

For the proof we need an analogous result for potentials, which can be considered as a generalization of a theorem by Littlewood[1] for subharmonic functions.

Theorem 3.2: *Let $F(e)$ be a completely additive function of a set, defined in $|z|<1$, and put*

$$m(r) = \int\int_{|\zeta|<r} |dF(\zeta)|, \quad r<1.$$

If $G(z;\zeta)$ denotes Green's function for $|z|<1$, singular at $z=\zeta$, and we assume

$$\int_0^1 \frac{m(r)}{(1-r)^\alpha}\, dr < \infty, \quad 0<\alpha<1,$$

$$m(r) = O(1), \quad \alpha = 1,$$

then

$$\lim_{r=1-0} \int\int_{|\zeta|<1} G(re^{i\Theta};\zeta)\, dF(\zeta) = 0 \tag{3.3}$$

holds true except perhaps when Θ belongs to a set E such that $L_{1-\alpha}(E)=0$, $0<\alpha<1$, and $\bar{C}(E)=0$, $\alpha=1$.

We shall give a detailed proof only for $0<\alpha<1$, the limit case $\alpha=1$ being quite simple.

As $G(z;\zeta)$ is positive, we may suppose without loss of generality that F *is non-negative.* We assert that for every Θ with

$$H(\Theta) \equiv \int_{\frac{1}{2}}^1 dr \int\int_{\frac{1}{2}<|\zeta|<1} \left|\frac{\partial G}{\partial r}(re^{i\Theta};\zeta)\right| dF(\zeta) < \infty$$

(3.3) holds true. For if we set, $\varrho<1$,

$$U = (U-U_\varrho) + U_\varrho = \int\int_{|\zeta|\le\varrho} + \int\int_{|\zeta|>\varrho} G(re^{i\Theta};\zeta)\, dF(\zeta),$$

$U-U_\varrho$ tends uniformly to zero, as $r\to 1$. Let r_0 be a fixed number less than 1. Then

[1] Littlewood [1].

$$\left|\overline{\lim_{r=1-0}} U(r e^{i\Theta}) - U_\varrho(r_0 e^{i\Theta})\right| = \left|\overline{\lim_{r=1-0}} U_\varrho(r e^{i\Theta}) - U_\varrho(r_0 e^{i\Theta})\right| \leq$$

$$\leq \int_{r_0}^{1} d r \int\int_{|\zeta|>\varrho} \left|\frac{\partial G}{\partial r}(r e^{i\Theta};\ \zeta)\right| d F(\zeta).$$

If we let $\varrho \to 1$ in this inequality, we get

$$0 \leq \overline{\lim_{r=1-0}} U(r e^{i\Theta}) \leq \int_{r_0}^{1} d r \int\int_{\frac{1}{2}<|\zeta|<1} \left|\frac{\partial G}{\partial r}(r e^{i\Theta};\ \zeta)\right| d F(\zeta),$$

and now letting $r_0 \to 1$, we obtain the desired result.

The function $H(\Theta)$ is lower semi-continuous. Thus the set E_p, where $H(\Theta) > p$, is open. Let μ be a set function belonging to Γ_h, $h(t) = t^{1-\alpha}$. Then

$$p\,\mu(E_p) \leq \int_{E_p} d\,\mu(\Theta) \int_{\frac{1}{2}}^{1} d r \int\int_{\frac{1}{2}<|\zeta|<1} \left|\frac{\partial G}{\partial r}(r e^{i\Theta};\zeta)\right| d F(\zeta).$$

Observing

$$G(z;\ \zeta) = \log\left|\frac{1 - z\bar{\zeta}}{z - \zeta}\right|,$$

we easily obtain, if $z = r e^{i\Theta}$ and $\zeta = \varrho e^{i\varphi}$,

$$\left|\frac{\partial G}{\partial r}(z;\ \zeta)\right| \leq \frac{1-\varrho^2}{|z-\zeta|\,|1-z\bar{\zeta}|}$$

and get

$$p\,\mu(E_p) \leq \int\int_{\frac{1}{2}<|\zeta|<1} (1-\varrho^2)\, d F(\zeta) \int_{\frac{1}{2}}^{1} d r \int_{E_p} \frac{d\,\mu(\Theta)}{|z-\zeta|\,|1-z\zeta|}.$$

If, in order to estimate the last integral, we now set $\eta = |\varrho - r|$, we find

$$\int_{E_p} \frac{d\,\mu(\Theta)}{|z-\zeta|\cdot|1-z\bar{\zeta}|} \leq \frac{A}{1-r\varrho} \int_0^{2\pi} \frac{d\,\mu(t,\varphi)}{t+\eta} \leq \frac{A}{1-r\varrho} \int_0^{2\pi} \frac{t^{1-\alpha}\, d t}{(t+\eta)^2}$$

and

$$\int_{E_p} \frac{d\mu(\Theta)}{|z-\zeta|\cdot|1-z\bar{\zeta}|} \leq \frac{A}{(1-r\varrho)\,|r-\varrho|^{\alpha}},$$

where A as in the sequel denotes certain constants. Integrating this inequality with respect to r, we obtain after some simple calculations

$$\int_{\frac{1}{2}}^{1} dr \int_{E_p} \frac{d\mu(\Theta)}{|z-\zeta|\cdot|1-z\bar{\zeta}|} \leq A\cdot(1-\varrho)^{-\alpha}$$

and at last

$$p\,\mu(E_p) \leq A \int_{\frac{1}{2}\leq|\zeta|<1} (1-\varrho)^{1-\alpha}\, dF(\zeta) \leq A \int_{\frac{1}{2}}^{1} \frac{m(\varrho)}{(1-\varrho)^{\alpha}}\, d\varrho.$$

Thus

$$L^{*}_{1-\alpha}(E_p) = \sup_{\mu\in\Gamma_h} \mu(E_p) \leq \frac{A}{p}.$$

From theorems 1.1 and 1.2 we then conclude that

$$\lim_{p=\infty} L_{1-\alpha}(E_p) \leq \lim_{p=\infty} \frac{A}{p} = 0,$$

and the theorem is proved.

We observe that theorem 3.2 implies that if $B(z)\in B_{\alpha}$, then $|B(re^{i\Theta})|\to 1$, $r\to 1$, a.e. with respect to $L_{1-\alpha}(E)$, but we cannot conclude that the argument tends to a limit a.e., and thus theorem 2.3 is not included in it.

16. We shall now use the above to prove the theorem concerning radial limits for functions in T_{α} mentioned earlier.

Theorem 3.3: *If* $w(z)\in T_{\alpha}$, $0<\alpha\leq 1$, *then* $\lim_{r=1-0} w(re^{i\Theta})$ *exists for every* Θ *outside a set whose outer* $(1-\alpha)$*-capacity vanishes.*

Suppose that $a_0 \neq w(0)$ is an α-regular value[1] of $w(z)$ and put

$$f(z) = \frac{1+w(z)\,\bar{a}_0}{w(z)-a_0}. \tag{3.4}$$

[1] See page 20.

Then $f(z)$ belongs to T_a and ∞ is α-regular, since the transformation (3.4) is a rotation of the sphere. We now take $c = \gamma + i\delta$, $\gamma\delta \neq 0$, and set

$$g_c(z) = g(z) = f(z) - c.$$

We consider the auxiliary function

$$u(z) = \{\log(2 + |g(z)|^2)\}^s, \quad 0 < s < \frac{1}{2}. \tag{3.5}$$ [1]

Then

$$|\Delta u(z)| = \Psi(|g(z)|)\,|g'(z)|^2,$$

where Δ is Laplace's operator and

$$\Psi(t) \sim \frac{A(s)}{t^2(\log t)^{2-s}}, \quad t \to \infty, \tag{3.6}$$

and $\Psi(t)$ is bounded for all t. We use Green's formula

$$\int_\Gamma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds = \iint_D (u\,\Delta v - v\,\Delta u)\, d\sigma_\zeta, \tag{3.7}$$

which is valid if $u(\zeta)$ and $v(\zeta)$ are continuous and have continuous derivatives of the first two orders in D and on its boundary Γ; $\frac{\partial}{\partial n}$ denotes derivation along the outer normal. Let R be a fixed number less than 1. We use (3.7) with

$$\begin{cases} u = u(R\zeta) \\ v = G(z;\zeta) = \log\left|\dfrac{1 - z\bar{\zeta}}{z - \zeta}\right|. \end{cases}$$

For D we take the domain bounded by the unit circle and small circles around z and the poles of $f(R\zeta)$. Letting the radii of these circles tend to zero, we obtain

$$u(Rz) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - r^2)\,u(Re^{i\varphi})}{1 + r^2 - 2r\cos(\Theta - \varphi)}\, d\varphi - \iint_{|\zeta|<1} \frac{\Delta u(R\zeta)}{2\pi} G(z;\zeta)\, d\sigma_\zeta,$$

[1] If we only wanted to prove the present theorem, we could choose $u = [1 + |g(z)|^2]^{-1}$ instead, which would be somewhat simpler; using (3.5), however, we obtain theorem 3.4 also.

where $z = r e^{i\Theta}$. Using obvious notations, we write

$$u(Rz) = u_R(z) - U_R(z),$$

where $u_R(z)$ is a harmonic function and $U_R(z)$ a potential. As $R \to 1$, $U_R(z)$ tends to a limit, finite or infinite, for

$$U_1(z) = \lim_{R=1-0} U_R(z) = \iint_{|\zeta|<1} \frac{\Delta u(\zeta)}{2\pi} G(z;\zeta)\, d\sigma_\zeta = \iint_{|\zeta|<1} G(z;\zeta)\, dF(\zeta).$$

$U_1(z)$ satisfies the conditions of theorem 3.2 since

$$m(r) = \iint_{|\zeta|<r} |dF(\zeta)| = \frac{1}{2\pi} \iint_{|\zeta|<r} |\Delta u(\zeta)|\, d\sigma_\zeta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(|a|)\, n(r,a)\, d\sigma_a,$$

where $n(r,a) = n_c(r,a)$ has its usual meaning, and, $0 < \alpha < 1$,

$$2\pi \int_0^1 \frac{m(r)}{(1-r)^\alpha}\, dr = \int_{-\infty}^{\infty} \Psi(|a|)\, d\sigma_a \int_0^1 \frac{n(r,a)}{(1-r)^\alpha}\, dr. \tag{3.8}$$

For if we observe (3.6) and the fact that ∞ is α-regular, we easily realize that the integrals in (3.8) are convergent. We thus obtain that

$$\lim_{r=1-0} U_1(r e^{i\Theta}) = 0$$

outside a set of outer $(1-\alpha)$-capacity zero.

We now turn to the harmonic functions $u_R(z)$. By Harnack's principle, these functions have a limit function, harmonic in $|z|<1$, as $R \to 1$:

$$u_R(z) = \frac{a_0(R)}{2} + \sum_1^\infty [a_n(R)\cos n\Theta + b_n(R)\sin n\Theta]\, r^n, \quad R \le 1, \quad r < 1.$$

We assert that

$$\sum_1^\infty [a_n(R)^2 + b_n(R)^2]\, n^\alpha \le M \tag{3.9}$$

independently of $R<1$. We have

$$\left(\frac{\partial u(Rz)}{\partial r}\right)^2 + \left(\frac{\partial u(Rz)}{\partial \Theta}\right)^2 \cdot \frac{1}{r^2} \leq \frac{8 s^2 \cdot |g'(z)|^2 \cdot |g(z)|^2}{(2+|g(z)|^2)^2 \{\log(2+|g(z)|^2)\}^{2-2s}}.$$

In the same way as before we find, using the notations in theorem 3.1, that

$$D_\alpha(u(Rz)) \leq M', \quad R<1,$$

and conclude that (3.9) holds true. The same inequality is now valid for $R=1$, and we can use the theorem concerning trigonometrical series, cited in nr. 9, to deduce that

$$\lim_{r=1-0} u_1(r e^{i\Theta})$$

exists and is finite outside a set of outer $(1-\alpha)$-capacity zero. Combining the results, we have proved that

$$\lim_{r=1-0} |f(r e^{i\Theta}) - c|^2, \quad c = \gamma + i\delta, \tag{3.10}$$

exists finitely outside such a set. If we make the same argument for the four different values of c, $\pm\gamma \pm i\delta$, and subtract the corresponding expressions (3.10) two by two, our proposition is found to be completely proved.

As, however, the limit (3.10) was finite under the assumption that a_0 in (3.4) was α-regular, we can in (3.4) let a_0 run through all α-regular values.[1] In this way we obtain the following theorem.

Theorem 3.4: *If* $w(z) \in T_\alpha$, $0<\alpha\leq 1$, *and* E_a *is the set of* Θ*-values for which* $\lim_{r=1-0} w(r e^{i\Theta}) = a$, *then* $\overline{C}_{1-\alpha}(E_a) = 0$ *except perhaps when* $a \in Q_a$ *and thus for almost all* a *on the sphere.*

17. The characteristic property of a meromorphic function belonging to T_α is expressed by a restriction on the growth of the area of its Riemann surface. It is interesting that the theorem 3.3 is valid even if the area has an arbitrarily rapid growth over a totally disconnected set E. It is possible to prove the following more general theorem, which has been communicated to me by professor Beurling in the case $\alpha = 1$.

Theorem 3.5: *Let* $w(z)$ *be meromorphic in* $|z|<1$ *and* O *be the set of points on the sphere such that for every* $w \in O$ *there exists a calotte* $C_w: [w; a] < \delta(w)$, *having the property that*

[1] $w(0) \neq a_0$ is evidently inessential.

$$\int\int_{C_w} \frac{d\sigma_a}{(1+|a|^2)^2} \int_0^1 \frac{n(r, a)}{(1-r)^\alpha}\, dr < \infty, \quad 0 < \alpha < 1,$$

$$\int\int_{C_w} \frac{d\sigma_a}{(1+|a|^2)^2}\, n(1, a) < \infty, \quad \alpha = 1.$$

For every Θ outside a set of outer $(1-\alpha)$-capacity zero it then holds true that either

(i) $\lim_{r=1-0} w(re^{i\Theta})$ *exists*

or

(ii) *the cluster-set of $w(re^{i\Theta})$, as $r \to 1$, is a continuum in the complement E of O.*

Corollary: *If E is totally disconnected, then* $\lim_{r=1-0} w(re^{i\Theta})$ *exists except perhaps on a set of outer $(1-\alpha)$-capacity zero.*

The set E is evidently closed. We can cover O by a countable family F of calottes C_w. Let C be one of these. We choose a function $\varphi(w)$ in the following way:

(a) $\varphi(w) = 0$ outside C and > 0 inside C.

(b) $\varphi(w)$ has continuous derivatives of the first two orders on the whole sphere.

(c) $\varphi(w) = \lambda > 0$ is valid only on a smooth curve.

It is now easily seen that we can use the same argument as above on the function $\varphi(w(re^{i\Theta}))$ and we conclude that

$$\lim_{r=1-0} \varphi(w(re^{i\Theta}))$$

exists outside a set of outer $(1-\alpha)$-capacity zero. If this limit is positive and (i) does not hold, then $w(re^{i\Theta})$ has to approximate values on a level-curve $\varphi = \lambda$, as $r \to 1$. We now choose a function φ_1 of the same kind as φ but so that every level-curve $\varphi_1 = \lambda_1 > 0$ has only isolated points in common with every curve $\varphi = \lambda > 0$, and use the same argument on φ_1. If for a Θ-value φ and φ_1 tend to positive limits, then (i) holds, because of the construction of φ_1. Thus, outside a set of $(1-\alpha)$-capacity zero, either

(i') $\lim_{r=1-0} w(r e^{i\Theta})$ exists

or

(ii') the cluster-set at $e^{i\Theta}$ lies outside C.

Since C is any calotte in the family F and a countable union of null-sets is a null-set, our theorem is completely proved.

§ 3. A QUOTIENT THEOREM

18. The theorems (B) and (C) in the introduction, to which our results above correspond, are generally proved with the aid of the quotient theorem (A). It may seem natural to use the same method here, i.e. to prove that any function in T_α is the quotient of two bounded functions in T_α. However, there seems to be an essential difference between a bounded and an arbitrary function in T_α, $\alpha > 0$, with regard to the existence of boundary values; we have proved that for the class T_α the exceptional set has $(1-\alpha)$-capacity zero and that for the class B_α this set has vanishing Hausdorff measure of order $1-\alpha$. Observing the difference between these two measures, we can make the conjecture that the following theorem, which we shall prove for $\delta > 0$, is false for $\delta = 0$.

Theorem 3.6: *If* $w(z) \in T_\alpha$, $0 < \alpha \leq 1$, *there exist two bounded functions*

$$\varphi(z) = \sum_0^\infty a_n z^n, \quad \psi(z) = \sum_0^\infty b_n z^n$$

such that

$$w(z) = \frac{\varphi(z)}{\psi(z)}$$

and

$$\sum_2^\infty |a_n|^2 \frac{n^\alpha}{(\log n)^{1+\delta}} < \infty, \quad \sum_2^\infty |b_n|^2 \frac{n^\alpha}{(\log n)^{1+\delta}} < \infty$$

for every $\delta > 0$. *In particular,* φ *and* ψ *belong to every class* T_β *with* $\beta < \alpha$.

To a given real integrable function $g(\Theta)$ we associate the analytic function

$$g(z)+i\,\tilde{g}(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,g(\varphi)\,d\varphi,$$

and denote a.e. its boundary values $g(\Theta)+i\,\tilde{g}(\Theta)$. Then the following lemma holds.

Lemma: *If*

$$R(x)=\begin{cases} x^2, & |x|\leq 1,\\ |x|^{\frac{3}{2}}, & |x|\geq 1,\end{cases}$$

then there exists a numerical constant C so that

$$\int_0^{2\pi} R(\tilde{g}(\Theta))\,d\Theta\leq C\cdot\int_0^{2\pi} R(g(\Theta))\,d\Theta.$$

We define $R_1(x)$ to be equal to $R(x)$ except for $1<|x|<6$, where it is defined so that R_1'' exists and is continuous and positive everywhere. By use of the well-known Stein method for conjugate functions [1] we obtain

$$\int_0^{2\pi} R_1(\tilde{g}(\Theta))\,d\Theta\leq C_1\int_0^{2\pi} R_1(g(\Theta))\,d\Theta,$$

and the lemma is an immediate consequence of this.

To prove the theorem we next make an auxiliary transformation and put

$$f(z)=\frac{w(z)-w_0}{1+w(z)\,\overline{w}_0}.$$

We can choose the number w_0 so that $f(z)$ has the following properties:

(a) If α_ν and β_ν, $\nu=1,2,\ldots$, are the zeros, respectively poles, of $f(z)$, the series $\sum_1^\infty(1-|\alpha_\nu|)^{1-\alpha}$ and $\sum_1^\infty(1-|\beta_\nu|)^{1-\alpha}$ converge.

[1] See Zygmund [1], pages 149 and 165, ex. 6.

(b) 0 and ∞ are a-regular values for the function $f(z)$.

(c)

$$\lim_{\varrho=\overline{1-0}} \int_0^{2\pi} |f(\varrho\, e^{i\Theta})|\, d\,\Theta < \infty$$

$$\lim_{\varrho=\overline{1-0}} \int_0^{2\pi} \frac{1}{|f(\varrho\, e^{i\Theta})|}\, d\,\Theta < \infty\,.$$

This choice is possible because (a) and (b) are each satisfied by a.a. w_0 (See theorem 4.2); by theorem 1.6 the same is true for (c). We may also suppose that $f(0) \neq 0, \infty$, and set

$$\begin{cases} A(z) = \prod_1^{\infty} \dfrac{a_\nu - z}{1 - z\,\bar{a}_\nu} \cdot \dfrac{\bar{a}_\nu}{|a_\nu|} \\ B(z) = \prod_1^{\infty} \dfrac{\beta_\nu - z}{1 - z\,\bar{\beta}_\nu} \cdot \dfrac{\bar{\beta}_\nu}{|\beta_\nu|} \end{cases}$$

These functions are bounded and belong to T_a on account of theorem 2.2.

After these preliminaries, we consider the function

$$g_\varrho(\Theta) = \overset{+}{\log} |f(\varrho\, e^{i\Theta})| + 1,\ \varrho < 1,$$

and set

$$G_\varrho(z) = e^{-g_\varrho(z) - i\,\tilde{g}_\varrho(z)} = \sum_0^{\infty} c_n(\varrho)\, z^n.$$

Then $G_\varrho(z)$ is bounded in $|z| < 1$. We cannot conveniently work with $g_\varrho(\Theta)$ but have to consider a function $u_\varrho(\Theta)$ of slower growth:

$$[g_\varrho(\Theta)]^s = u_\varrho(\Theta) \sim \frac{a_0(\varrho)}{2} + \sum_1^{\infty} (a_n(\varrho)\cos n\Theta + b_n(\varrho)\sin n\Theta),$$

where $s < \frac{1}{2}$ is a number that will be fixed later on. From theorem 3.1 and the property (b) above it follows in the same way as in the proof of theorem 3.3 that

$$\sum_{1}^{\infty} [a_n(\varrho)^2 + b_n(\varrho)^2]\, n^a \leq M_1$$

independently of ϱ. Introducing the abbreviation

$$Q(\varphi;\ \Theta, t) = \varphi(\Theta + t) - \varphi(\Theta - t),$$

we conclude from (2.5) that

$$S_{a,0}(u_\varrho) = \int_0^{\frac{1}{2}} \frac{dt}{t^{1+a}} \int_0^{2\pi} |Q(u_\varrho;\ \Theta, t)|^2\, d\Theta \leq M_2 .$$

Since $\bar{g}_\varrho(\Theta+t) - \bar{g}_\varrho(\Theta-t)$ as a function of Θ is conjugate to $g_\varrho(\Theta+t) - g_\varrho(\Theta-t)$, it follows from the lemma that

$$\int_0^{2\pi} R(Q(\bar{g}_\varrho;\ \Theta, t))\, d\Theta \leq C \int_0^{2\pi} R(Q(g_\varrho;\ \Theta, t))\, d\Theta =$$

$$= C \int_0^{2\pi} R(Q(u_\varrho^{\frac{1}{s}};\ \Theta, t))\, d\Theta \leq \tag{3.11}$$

$$\leq \frac{2C}{s^2} \int_0^{2\pi} R(Q(u_\varrho;\ \Theta, t)) \left[u_\varrho(\Theta+t)^{\frac{2}{s}-2} + u_\varrho(\Theta-t)^{\frac{2}{s}-2} \right] d\Theta ,$$

if we observe that

$$R(x^p - y^p) \leq p^2\, R(x-y)\, [x^{p-1} + y^{p-1}]^2$$

for $p, x, y \geq 1$. If $\beta > 1$, we use in the integral

$$I(\varrho) = \int_0^{\frac{1}{2}} \frac{dt}{t^{1+a} \left(\log \frac{1}{t} \right)^{\beta}} \int_0^{2\pi} R(Q(u_\varrho;\ \Theta, t))\, u_\varrho(\Theta+t)^{\frac{2}{s}-2}\, d\Theta$$

the inequality of Young,

$$x \cdot y \leq \Phi(x) + \Psi(y)$$

with

$$\Phi(x) = x\,(\overset{+}{\log} x)^{\beta}$$

and

$$\Psi(y) = K_\gamma \cdot e^{y^\gamma}, \quad \frac{1}{\beta} < \gamma < 1,$$

for a suitable constant K_γ. We obtain

$$I(\varrho) \leq I_1(\varrho) + I_2(\varrho) = \int_0^{\frac{1}{2}} \int_0^{2\pi} \Phi\left\{ \frac{R(Q)}{t^{1+\alpha}\left(\log \frac{1}{t}\right)^{\beta}} \right\} d\Theta\, dt +$$

$$+ \int_0^{\frac{1}{2}} \int_0^{2\pi} \Psi\left\{ u_\varrho\,(\Theta + t)^{\frac{2}{s}-2} \right\} d\Theta\, dt.$$

Recalling the definitions of Φ and R and observing that if $|Q| \leq 1$, $\overset{+}{\log} R(Q) = 0$, while if $|Q| > 1$, $|Q|^{\frac{3}{2}} (\log |Q|)^{\beta} < A(\beta)\,|Q|^2$, we find that

$$I_1(\varrho) \leq A(\alpha, \beta) \int_0^{\frac{1}{2}} \frac{dt}{t^{1+\alpha}} \int_0^{2\pi} |Q(u_\varrho; \Theta, t)|^2\, d\Theta \leq M_3$$

independently of ϱ. Furthermore

$$I_2(\varrho) = K_\gamma \int_0^{\frac{1}{2}} \int_0^{2\pi} e^{g_\varrho (\Theta + t)^{\gamma(2-2s)}}\, d\Theta\, dt.$$

By taking s so that

$$\gamma(2 - 2s) = 1$$

we get the estimation

$$I_2(\varrho) \leq \frac{e}{2} K_\gamma \int_0^{2\pi} (|f(\varrho\, e^{i\Theta})| + 1)\, d\Theta.$$

In starting with the term in (3.11) containing $u_\varrho(\Theta - t)$, we obtain the same estimations, whence, on account of our assumption (c),

$$\lim_{\varrho=\overline{1-0}} \int_0^{\frac{1}{2}} \frac{dt}{t^{1+\alpha}\left(\log\frac{1}{t}\right)^{\beta}} \int_0^{2\pi} R\left(Q\left(\bar{g}_\varrho;\Theta,t\right)\right) d\Theta < \infty.$$

Because of the definition of R, we find

$$\lim_{\varrho=\overline{1-0}} S_{\alpha,\beta}\left(e^{-i\bar{g}_\varrho(\Theta)}\right) < \infty.$$

Observing that

$$|Q\left(G_\varrho(e^{i\Theta});\Theta,t\right)| \leq |Q\left(e^{-g_\varrho}\right)| + |Q\left(e^{-i\bar{g}_\varrho}\right)| \leq \tag{3.12}$$

$$\leq A(s)\,|Q(u_\varrho)| + |Q\left(e^{-i\bar{g}_\varrho}\right)|,$$

we get

$$\lim_{\varrho=\overline{1-0}} S_{\alpha,\beta}\left(G_\varrho(e^{i\Theta})\right) < \infty$$

and on account of (2.5)

$$\lim_{\varrho=\overline{1-0}} \sum_2^{\infty} |c_n(\varrho)|^2 \frac{n^\alpha}{(\log n)^\beta} < \infty. \tag{3.13}$$

Now it is known that[1]

$$\lim_{\varrho=1-0} G_\varrho(z) = G_1(z)$$

exists uniformly for $|z| \leq r_0 < 1$. We thus obtain by (3.13) and by another use of (2.5) that

$$S_{\alpha,\beta}\left(G_1(e^{i\Theta})\right) < \infty. \tag{3.14}$$

Making the same argument for $\frac{1}{f(z)}$, which is possible on account of our assumptions (b) and (c), we obtain a corresponding limit function $H_1(z)$ having the same property (3.14). We get at last[1]

$$f(z) = \frac{w(z) - w_0}{1 + w(z)\,\overline{w}_0} = \frac{A(z)}{B(z)} \cdot \frac{H_1(z)}{G_1(z)} \cdot e^{i\varphi}.$$

[1] Nevanlinna [1], page 174 ff.

Solving for $w(z)$, we have a representation of $w(z)$ as a quotient of linear combinations of functions whose limit functions on $|z|=1$ all have the $S_{\alpha,\beta}$-property (3.14). By the distribution argument (3.12) and (2.5) our proposition is thus completely established.

19. In this proof it was very important that $|f(\varrho e^{i\Theta})|$ was not too large on a certain sequence of circles with radii tending to 1. It is natural to ask if in the T_a-case anything more than

$$\int_0^{2\pi} \overset{+}{\log} \left| \frac{1}{w(r e^{i\Theta}) - a} \right| d\Theta = O(1)$$

for *all* r is true. We can prove the following theorem.

Theorem 3.7: *If* $w(z) \in T_a$, $0<\alpha<1$, *then for every* a *not belonging to the singular set* Q_a *and thus for almost all* a

$$\int_0^{2\pi} \left\{ \overset{+}{\log} \left| \frac{1}{w(r e^{i\Theta}) - a} \right| \right\}^{\sigma} d\Theta = O(1), \quad r \to 1, \tag{3.15}$$

for every $\sigma < \dfrac{1}{1-\alpha}$.

If $w(z) \in T_1$, *then*

$$\overline{\lim_{r=1-0}} \int_0^{2\pi} \frac{d\Theta}{|w(r e^{i\Theta}) - a|^p} < \infty \tag{3.16}$$

for almost all a, *if* $p<2$.

The proof depends on the following lemma which is due to Hardy-Littlewood[1].

Lemma: *If*

$$\varphi(x) \sim \sum_1^{\infty} (a_n \cos n x + b_n \sin n x),$$

then

---

[1] Hardy-Littlewood [1], page 199; Zygmund [1], page 234, ex. 3.

$$(3.17)\qquad \left\{\int_0^{2\pi} |\varphi(x)|^q\, dx\right\}^{\frac{1}{q}} \leq A_\alpha \left\{\sum_1^\infty (a_n^2+b_n^2)\, n^\alpha\right\}^{\frac{1}{2}}, \quad 0\leq\alpha<1,$$

where $q=\dfrac{2}{1-\alpha}$ *and* A_α *only depends on* α.

To prove our theorem we need only observe that, by theorem 3.1, for any α-regular a the Fourier coefficients of

$$\left\{\overset{+}{\log}\left|\frac{1}{w(re^{i\Theta})-a}\right|\right\}^s, \quad 0<s<\frac{1}{2},$$

are such that the corresponding series in (3.17) is uniformly bounded in r. We note for later applications that if

$$\int_0^1 \frac{n(r,a)}{(1-r)^\alpha}\, dr \leq A$$

for all a, then (3.15) is bounded uniformly in a as $r\to 1$.

As to the case $\alpha=1$, any function belonging to T_1 takes almost all values no more than a finite number of times. Thus, for a not belonging to a certain set E, $m_2 E=0$, the integrand in (3.16) is subharmonic for $r_0(a)<r<1$, and we conclude that the integral is a convex and continuous function of $\log r$ in $r_0<r<1$. Hence the limit (3.16) exists for these a's, and we can replace the upper limit with the lower limit. An application of theorem 1.6 proves the theorem.

Finally we remark that (3.15) need not hold for *every* a as the simple example

$$\begin{cases} w(z)=e^{\frac{z+1}{z-1}} \\ a=0 \end{cases}$$

shows. See below, page 75.

CHAPTER IV

ON THE DISTRIBUTION OF VALUES FOR FUNCTIONS IN T_α

20. If $w(z)$ is meromorphic in $|z|<1$ and the characteristic function $T(r)$ is unbounded as $r\to 1$, then it is known that there may only exist a very small set of exceptional values a, having the property of being taken comparatively rarely by the function $w(z)$. When $T(r)$ is bounded, i.e. when $w(z)$ belongs to T_0, the picture is complety changed, and a function can omit a very large set without belonging to any class T_α with $\alpha>0$, as the simple instance

$$w(z)=\sum_1^\infty \frac{1}{n^2} z^{2^n}$$

shows. The exceptional values for these classes T_α turn out to be those which are taken *frequently*. To get a measure of how frequently a value is taken, we associate the series

$$\sum_1^\infty (1-r_\nu(a))^{1-\gamma},\quad 0\le\gamma\le\alpha\,, \tag{4.1}$$

with a function $w(z)\in T_\alpha$, where $r_\nu(a)$, $\nu=1,2,\ldots$, are the moduli of the zeros of $w(z)-a$. We are going to study the set S_γ where the series (4.1) diverges.

First, however, without making any pretension to generality, we shall prove the following theorem, showing that the series $\sum_1^\infty (1-r_\nu(a))^{1-\gamma}$, $\gamma>0$, may diverge for functions belonging to T_α.

Theorem 4.1: *Let* $\{a_\nu\}_1^\infty$ *be a sequence of real numbers,* $0<a_\nu<1$, *for which* $\sum_1^\infty (1-a_\nu)$ *converges. Then the Blaschke product*

$$B(z)=\prod_1^\infty \frac{a_\nu - z}{1-z\,a_\nu}=\sum_1^\infty C_n z^n$$

belongs to every class T_α *with* $\alpha<\frac{1}{2}$.

Since $|B(r e^{i\Theta})| \to 1$, $r \to 1$, if $\Theta \not\equiv 0 \pmod{2\pi}$, we have

$$\sum_0^\infty |C_n|^2 = 1 .$$

Furthermore, we have, if $z = r e^{i\Theta}$, $0 \leq \Theta \leq \pi$.

$$|B(r e^{i\Theta})|^2 = \prod_1^\infty \left(1 - \frac{(1-r^2)(1-a_\nu^2)}{|1 - z\,a_\nu|^2}\right) > 1 - \sum_{\nu=1}^\infty \frac{(1-r^2)(1-a_\nu^2)}{|1 - z\,a_\nu|^2},$$

and we easily conclude that there exists a constant A so that

$$1 - |B(r e^{i\Theta})|^2 < A \frac{\eta}{\eta^2 + \Theta^2}, \quad \eta = 1 - r > 0,$$

and thus

$$\int_0^{2\pi} (1 - |B(r e^{i\Theta})|^2)\, d\Theta < 2 \int_0^{\sqrt{\eta}} d\Theta + 2A \int_{\sqrt{\eta}}^{\pi} \frac{\eta\, d\Theta}{\eta^2 + \Theta^2}.$$

Inserting the Taylor series, we get

$$2\pi \sum_1^\infty |C_n|^2 (1 - r^{2n}) < 2\sqrt{\eta} + 2A \int_{1/\sqrt{\eta}}^\infty \frac{dt}{1+t^2} < (2A+2)\sqrt{\eta}.$$

If, thus, α is a positive number less than $\frac{1}{2}$

$$\int_0^1 \frac{dr}{(1-r)^{1+\alpha}} \sum_1^\infty |C_n|^2 (1 - r^{2n}) < \infty.$$

But since

$$\int_0^1 \frac{1 - r^{2n}}{(1-r)^{1+\alpha}}\, dr = -\frac{1}{\alpha} + \frac{2n}{\alpha} \int_0^1 \frac{r^{2n-1}}{(1-r)^\alpha}\, dr \sim c \cdot n^\alpha ,$$

where c is a positive constant, we find at last

$$\sum_1^\infty |C_n|^2 n^\alpha < \infty,$$

and the theorem is proved.

As the series $\sum_1^\infty (1-a_\nu)$ may converge arbitrarily slowly, we have in this way constructed a class of examples of the desired kind. We also observe that this theorem implies that the class B_a considered in Chapter II is not identical with the class of Blaschke products in T_a.

21. We now formulate our principal result concerning the distribution of values of a function in T_a.

Theorem 4.2: *If* $w(z) \in T_a$ *and* S_γ, $0 \leq \gamma \leq a$, *is the set where* $\sum_1^\infty (1-r_\nu(a))^{1-\gamma}$ *diverges, then*

(a) S_0 *is empty,*

(b) Dim $(S_\gamma) \leq \frac{2\gamma}{a}$, $0 < \gamma < a$,

(c) $m_2 S_a = 0$.

We observe that there is "continuous relationship" between the three cases (a), (b) and (c).

(a) and (c) are quite trivial. (a) is nothing but the theorem (D) cited in the introduction and (c) is an immediate consequence of the representation (2.1) of $T_a(w)$, if we observe that, $w(0) \neq a$,

$$\sum_1^\infty (1-r_\nu(a))^{1-\gamma} = (1-\gamma)\int_0^1 \frac{n(r,a)}{(1-r)^\gamma}\,dr.$$

In the proof of (b) we consider two different cases: (I) $0 < \gamma < \frac{a}{2}$ and (II) $\frac{a}{2} < \gamma < a$. The argument depends essentially on theorem 1.5.

We introduce a function $\varphi(w)$ such that

1°. $\varphi(w) = w$, $|w| < \frac{2}{3}$.

2°. $\varphi(w) = \frac{1}{\overline{w}}$, $|w| > \frac{3}{2}$.

3°. $|\varphi(w)| \geq \frac{2}{3}$, $\frac{2}{3} \leq |w| \leq \frac{3}{2}$, and is defined in this range so that it is continuous and has continuous derivatives of the first two orders in the whole plane.

We then consider the bounded function

$$\psi(z) = \varphi(w(z)) \sim \sum_{-\infty}^{\infty} a_n(r)\, e^{ni\Theta}, \quad z = r\, e^{i\Theta}.$$

There is a constant C so that

$$\left.\begin{matrix} \left|\frac{\partial \psi}{\partial x}\right| \\ \left|\frac{\partial \psi}{\partial y}\right| \end{matrix}\right\} \leq C \frac{|w'(z)|}{1+|w(z)|^2}, \quad z = x + iy, \tag{4.2}$$

and

$$|\Delta \psi| \leq C \frac{|w'(z)|^2}{(1+|w(z)|^2)^2}. \tag{4.3}$$

Furthermore

$$\Delta \psi(z) = 0, \text{ if } |\psi(z)| \leq \frac{2}{3}. \tag{4.4}$$

We shall need the following lemma in the proof.

Lemma 1: *If* $0 < \lambda \leq 2$ *and* $\lambda a - 2\eta > 0$, *then*

$$\int_0^{\varrho} \int_0^{2\pi} \frac{|\psi(\varrho\, e^{i\Theta}) - \psi(r\, e^{i\Theta})|^{\lambda}}{(1-r)^{1+\eta}}\, d\Theta\, dr = O(1), \quad \varrho \to 1. \tag{4.5}$$

As an application of Hölder's inequality shows, it is sufficient to prove the lemma for $\lambda = 2$. Observing (4.2) we find

$$|\psi(\varrho\, e^{i\Theta}) - \psi(r\, e^{i\Theta})|^2 \leq C^2 2 \left(\int_r^{\varrho} \frac{|w'(t\, e^{i\Theta})|}{1+|w(t\, e^{i\Theta})|^2}\, dt \right)^2 \leq$$

$$\leq 2\, \frac{C^2}{a} (1-r)^a \int_r^1 \frac{|w'|^2}{(1+|w|^2)^2} (1-t)^{1-a}\, dt.$$

The integral (4.5) is thus dominated by

$$A(a) \int_0^{\varrho} \frac{dr}{(1-r)^{1+\eta-a}} \int_0^{2\pi} d\Theta \int_0^1 \frac{|w'|^2}{(1+|w|^2)^2} (1-t)^{1-a}\, dt,$$

which is bounded under our assumptions.

To prove the theorem it is sufficient to show that the subset of S_γ which lies in the circle $|a|<\frac{1}{2}$ satisfies the inequality (b). We therefore consider only a-values in this circle.

Case (I): Let γ, $0<\gamma<\frac{\alpha}{2}$, and β, $\frac{2\gamma}{\alpha}<\beta<1$, be two fixed numbers and E_p the set of points a, $|a|<\frac{1}{2}$, for which

$$\int_0^1 \frac{n(r,a)}{(1-r)^\gamma}\,dr>p.$$

This set is readily seen to be open. (We suppose that it is not empty). Let μ be a distribution of the unit mass on E_p and set

$$u(w)=\int_{E_p}\log\frac{1}{|w-a|}\,d\mu(a).$$

If M_μ is the least number for which

$$|u(w)-u(w')|\leq M_\mu|w-w'|^\beta \tag{4.6}$$

holds for all w and w', then, by theorems 1.5 and 1.1, it is sufficient to prove that

$$\lim_{p=\infty}\inf_{\mu} M_\mu=\infty \tag{4.7}$$

in order to conclude that $L_\beta(S_\gamma)=0$.

Let $\nu(r,a)$ be the number of a-values of $\psi(z)$ in the circle $|z|\leq r$. We observe that

$$n(r,a)\leq\nu(r,a). \tag{4.8}$$

Green's formula (3.7) yields in the usual manner

(4.9)

$$\int_0^\varrho\frac{dr}{(1-r)^\gamma}\frac{d}{dr}\int_0^{2\pi}\log\frac{1}{|\psi(re^{i\Theta})-a|}\,d\Theta-\int_0^\varrho\frac{dr}{r(1-r)^\gamma}\iint_{|\zeta|<r}\Delta\log\frac{1}{|\psi(\zeta)-a|}\,d\sigma_\zeta=$$

$$=-2\pi\int_0^\varrho\frac{\nu(r,a)}{r(1-r)^\gamma}\,dr.$$

If we observe that

$$\Delta_\zeta \log |\psi(\zeta) - a| = |w'(\zeta)|^2 \Delta_w \log |\varphi(w) - a|,$$

and the definition of $\varphi(w)$, we see that the second integral to the left is uniformly bounded in ϱ and a. We use the symbol $O(\,)$ to indicate this independence. Integration by parts, multiplication by $d\mu(a)$ and integration over E_p yields:

$$\gamma \int_0^\varrho \frac{dr}{(1-r)^{1+\gamma}} \int_0^{2\pi} \{u(\psi(\varrho e^{i\Theta})) - u(\psi(r e^{i\Theta}))\}\, d\Theta + O(1) =$$

$$= -2\pi \int_{E_p} d\mu(a) \int_0^\varrho \frac{\nu(r, a)}{r(1-r)^\gamma}\, dr.$$

On account of (4.6) we find

$$2\pi \int_{E_p} d\mu(a) \int_0^\varrho \frac{\nu(r, a)}{r(1-r)^\gamma}\, dr \leq O(1) + \gamma M_\mu \int_0^\varrho \int_0^{2\pi} \frac{|\psi_\varrho - \psi_r|^\beta}{(1-r)^{1+\gamma}}\, d\Theta\, dr.$$

If we let $\varrho \to 1$ and use lemma 1, we obtain, observing (4.8),

$$p \leq O(M_\mu).$$

Thus (4.7) holds and we have proved that $L_\beta(S_\gamma) = 0$. Since β can be chosen arbitrarily close to $\frac{2\gamma}{\alpha}$, (I) is established.

Case (II): Having fixed γ, $\frac{\alpha}{2} < \gamma < \alpha$, and β, $\frac{2\gamma}{\alpha} < \beta < 2$, and introduced the set E_p as before, we consider an $f(w)$,

$$f(w) = \int_{E_p} \frac{d\mu(a)}{w - a},$$

such that

$$|f(w) - f(w')| \leq M_\mu |w - w'|^{\beta - 1}.$$

If

$$\lim_{p=\infty} \inf_{\mu} M_\mu = \infty,$$

we conclude by theorem 1.5 that $\overline{C}_\beta(S_\gamma) = 0$.

We first prove

Lemma 2: *If*

$$f(\psi(\varrho\, e^{i\Theta})) \sim \sum_{-\infty}^{\infty} d_n(\varrho)\, e^{ni\Theta},$$

then

$$\sum_{-\infty}^{\infty} |d_n(\varrho)|^2 |n|^\tau = O(M_\mu^2),$$

if $\tau = 2\gamma - \alpha$.

We have, setting $\sigma = \beta - 1$,

$$S_{\tau,0}(f(\psi_\varrho)) \leq M_\mu^2 \int_0^{1/2} \frac{dt}{t^{1+\tau}} \int_0^{2\pi} |\psi(\varrho\, e^{i(\Theta+t)}) - \psi(\varrho\, e^{i(\Theta-t)})|^{2\sigma}\, d\Theta \leq$$

$$\leq M_\mu^2 \{S_{\alpha,0}(\psi_\varrho)\}^\sigma \left\{ 2\pi \int_0^{1/2} \frac{dt}{t^q} \right\}^{1-\sigma}$$

by Hölder's inequality, if

$$q = [1 + \tau - \sigma(1+\alpha)] \frac{1}{1-\sigma} = 1 - \frac{\sigma\alpha - \tau}{1-\sigma}.$$

Since $\sigma\alpha - \tau = \alpha\beta - 2\gamma > 0$, the second parenthesis is finite. If we observe (4.2) and use theorem 3.1 on the real and imaginary parts of ψ_ϱ, we find

$$\sum_{-\infty}^{\infty} |a_n(\varrho)|^2 |n|^\alpha = O(1), \tag{4.10}$$

and the lemma follows from (2.5).

To prove the theorem we remark that

$$\frac{\partial}{\partial r} \log|\psi(r\, e^{i\Theta}) - a| = \mathbf{Re} \left\{ \frac{\dfrac{\partial \psi}{\partial r}}{\psi - a} \right\}$$

and deduce from (4.9) that

$$\text{Re}\left\{\int_0^{\varrho}\frac{dr}{(1-r)^{\gamma}}\int_0^{2\pi} f(\psi(re^{i\Theta}))\frac{\partial\psi}{\partial r}(re^{i\Theta})\,d\Theta\right\}+O(1)=$$

$$=2\pi\int_{E_p} d\mu(a)\int_0^{\varrho}\frac{\nu(r,a)}{r(1-r)^{\gamma}}\,dr, \tag{4.11}$$

since $f(\psi)$ is continuous. We shall prove that the integral to the left is $O(M_\mu)$. To this end we must use the property (4.3) of ψ. In order to be able to use Green's formula, we introduce the (complex) harmonic function

$$v(re^{i\Theta})=\sum_{-\infty}^{\infty} d_n(\varrho)\left(\frac{r}{\varrho}\right)^{|n|}e^{ni\Theta},\quad r<\varrho,$$

which is equal to $f(\psi(\varrho e^{i\Theta}))$ for $r=\varrho$, and write

$$\int_0^{2\pi} f(\psi_r)\cdot\psi_r'\,d\Theta=\int_0^{2\pi}\psi_r'(f(\psi_r)-f(\psi_\varrho))\,d\Theta+$$

$$+\int_0^{2\pi}\psi_r'(v_\varrho-v_r)\,d\Theta+\int_0^{2\pi}\psi_r' v_r\,d\Theta. \tag{4.12}$$

When we perform the integration with respect to r in (4.11) and use the division above, the first two terms are found to be $O(M_\mu)$. For the first term we get the estimation

$$\left|\int_0^{\varrho}\frac{dr}{(1-r)^{\gamma}}\int_0^{2\pi}\psi_r'(f(\psi_r)-f(\psi_\varrho))\,d\Theta\right|^2\le$$

$$\le\int_0^{\varrho}\int_0^{2\pi}|\psi_r'|^2(1-r)^{1-\alpha}\,d\Theta\,dr\cdot M_\mu^2\int_0^{\varrho}\int_0^{2\pi}\frac{|\psi_\varrho-\psi_r|^{2\sigma}}{(1-r)^{1+2\gamma-\alpha}}\,d\Theta\,dr=O(M_\mu^2)$$

by (4.2) and lemma 1. The second term can be treated similarly if we use lemma 2.

In the third term in (4.12) we use Green's formula (3.7) and obtain

$$\int_0^{2\pi} \psi_r' v_r \, d\Theta = \int_0^{2\pi} \psi_r v_r' \, d\Theta + \frac{1}{r} \iint_{|\zeta|<r} v \, \Delta \psi(\zeta) \, d\sigma_\zeta,$$

as v is harmonic. If we observe (4.3) and $|v| = O(M_\mu)$, we find that the r-integration of the second term above yields a quantity which is also $O(M_\mu)$. At last we write

$$\int_0^{2\pi} \psi_r v_r' \, d\Theta = \int_0^{2\pi} (\psi_r - \psi_\varrho) v_r' \, d\Theta + \int_0^{2\pi} \psi_\varrho v_r' \, d\Theta,$$

where the first term after the r-integration can be estimated in the same way as above by Schwarz's inequality and lemma 1. In the second we use Parseval's relation:

$$I(\varrho) = \frac{1}{2\pi} \int_0^\varrho \frac{dr}{(1-r)^\gamma} \int_0^{2\pi} \psi_\varrho v_r' \, d\Theta = \sum_{n \neq 0} d_n(\varrho) a_{-n}(\varrho) \frac{|n|}{\varrho^{|n|}} \int_0^\varrho \frac{r^{|n|-1}}{(1-r)^\gamma} \, dr =$$

$$= \sum_{-\infty}^{\infty} d_n(\varrho) a_{-n}(\varrho) O(|n|^\gamma).$$

Thus

$$|I(\varrho)|^2 \leq \sum_{-\infty}^{\infty} |a_n(\varrho)|^2 |n|^\alpha \cdot \sum_{-\infty}^{\infty} |d_n(\varrho)|^2 O(|n|^{2\gamma-\alpha}) = O(M_\mu^2)$$

by (4.10) and lemma 2. We have thus proved that

$$\int_{E_p} d\mu(a) \int_0^\varrho \frac{n(r,a)}{r(1-r)^\gamma} \, dr = O(M_\mu),$$

and the proposition follows in the same way as in Case (I).

With the aid of theorems 4.1 and 4.2 we can now construct an example, showing that the quotient of two functions in T_α may fail to belong to this class, $\alpha > 0$, even if the functions are bounded.

Let $B(z)$ be the Blaschke product of theorem 4.1 and set

$$b_\nu = \frac{a_\nu + a_{\nu+1}}{2}, \quad a_\nu < a_{\nu+1}, \quad \nu = 1, 2, \ldots,$$

and

$$B_1(z) = \prod_1^\infty \frac{b_\nu - z}{1 - z\, b_\nu}.$$

If

$$f(z) = \frac{B(z)}{B_1(z)}$$

and $\sum_1^\infty (1 - a_\nu)^{1-\gamma}$ diverges for every $\gamma > 0$, then $f(z)$ cannot belong to any class T_α with $\alpha > 0$, though $B(z)$ and $B_1(z)$ belong to every class T_β with $\beta < \frac{1}{2}$. For if we let z move from a_ν to $a_{\nu+1}$ along the real axis, $f(z)$ takes every real value a, since $f(x)$ is real and continuous, $x \neq b_\nu$, and has a pole at b_ν. Hence, if the set S_γ is associated with $f(z)$ as above,

$$\operatorname{Dim}(S_\gamma) \geq 1$$

for all $\gamma > 0$. From theorem 4.2 it then follows that $f(z)$ does not belong to any class T_α with $\alpha > 0$, which was our assertion.

22. It may be surprising that the sets S_γ are allowed to be so big. This depends no doubt on the fact that the range of $w(z)$ can be extremely complicated. If, however, we restrict this domain, much more can be said. An instance of this is given by the class $\mathfrak{U}_\alpha$. In this case the set S_α is very small, but it need not be empty, as was shown by theorem 4.1.

Theorem 4.3: *If $f(z)$ belongs to $\mathfrak{U}_\alpha$, $0 < \alpha < 1$, then $\overline{C}(S_\alpha) = 0$.*

Fix a number $\varrho < 1$ and let E_p be the open set of points a, having the properties

E_p:

$$\begin{cases} \int_0^1 \frac{n(r, a)}{(1 - r)^\alpha}\, dr > p \\ |a| < \varrho. \end{cases}$$

According to Green's formula (3.7), we get in the usual manner

$$\frac{n(r,a)}{r} = -\frac{d}{dr}\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{1-\bar{a}f(z)}{f(z)-a}\right| d\Theta, \quad z = re^{i\Theta}.$$

If μ denotes a distribution of the unit mass on E_p so that

$$\int_{E_p} \log\left|\frac{1}{w-a}\right| d\mu(a) \leq V_\mu < \infty,$$

we find by a simple argument

$$O(V_\mu) + \int_{E_p} d\mu(a) \int_0^1 \frac{n(r,a)}{r(1-r)^\alpha}\, dr =$$

$$= \frac{\alpha}{2\pi}\int_0^1 \frac{dr}{(1-r)^{1+\alpha}} \int_0^{2\pi} d\Theta \int_{E_p} \log\left|\frac{1-\bar{a}f}{f-a}\right| d\mu, \tag{4.13}$$

the integrals being convergent or divergent together, and where the symbol $O(\ )$ has the same sense as above. Let $H(r)$ be the set of Θ-values on which $f(re^{i\Theta})$ takes values in the circle $|f| \leq \frac{1+\varrho}{2}$. Since

$$\int_0^1 \frac{dr}{(1-r)^{1+\alpha}} \int_0^{2\pi} (1-|f(re^{i\Theta})|^2)\, d\Theta < \infty, \tag{4.14}$$

which is an immediate consequence of the definition of the class $\mathfrak{U}_\alpha$ (see page 48), it follows that

$$\int_0^1 \frac{m_1 H(r)}{(1-r)^{1+\alpha}}\, dr < \infty.$$

We write the second integral in (4.13): $I = I_1 + I_2$, where

$$I_1 = \int_0^1 \frac{dr}{(1-r)^{1+\alpha}} \int_{H(r)} d\Theta \int_{E_p} \log \left| \frac{1-\bar{a}f}{f-a} \right| d\mu <$$

$$< (\log 2 + V_\mu) \int_0^1 \frac{m_1 H(r)}{(1-r)^{1+\alpha}} dr = O(V_\mu)$$

and

$$I_2 = \int_0^1 \frac{dr}{(1-r)^{1+\alpha}} \int_{(0,2\pi)-H(r)} d\Theta \int_{E_\varrho} \log \left| \frac{1-\bar{a}f}{f-a} \right| d\mu <$$

$$< A(\varrho) \int_0^1 \frac{dr}{(1-r)^{1+\alpha}} \int_0^{2\pi} (1 - |f(re^{i\Theta})|^2)\, d\Theta = O(1)$$

on account of (4.14). We obtain

$$p = O(V_\mu)$$

for all μ and p, and we conclude that

$$\lim_{p=\infty} C(E_p) = 0.$$

The subset of S_a contained in $|a| < \varrho$ has thus outer logarithmic capacity zero. Letting $\varrho \to 1$ and observing that a countable union of null-sets is a null-set, we deduce our theorem.

CHAPTER V

ON COVERING SURFACES

23. Let G be a domain containing the point at infinity and bounded by a compact set F. Then there exists over G a Riemann surface G^∞, the universal covering surface, which is simply connected and which has the property that every function $z(w)$ which is analytic in G, is single-valued on G^∞. According to known theorems, if F contains more than two points, there exists a function $w(z; F)$, meromorphic in $|z|<1$, which represents this circle conformally on G^∞. $w(z; F)$ is uniquely determined by F up to a linear transformation of the unit circle into itself.

The theorem (E) of the introduction shows the remarkable relation existing between the class T_0 and the set F. When we study the classes T_α, however, we find a much more complicated situation and our results are very far from being complete. It will be shown that no purely metrical conditions on F are sufficient to make $w(z; F)$ belong to a given class T_α, $\alpha>0$. We will chiefly be concerned with the problem of finding the properties F will have when $w(z; F)$ belongs to T_α, and we observe that a necessary and sufficient condition for $w(z; F)$ to belong to T_1 is that G be simply connected.

24. We start with two general theorems which are of importance for the following. The first expresses the fact that if the connectivity of the corresponding domain is unchanged, then $T_\alpha(w(z; F))$ depends monotonically on F.

Theorem 5.1: *Let F and $\tilde{F}$ be two compact sets, $F \supset \tilde{F}$, G and $\tilde{G}$ the corresponding regions. We suppose that if Γ is any closed Jordan curve in G enclosing a point in F, it also encloses a point in $\tilde{F}$. In other words, we suppose that $\tilde{G}^\infty \doteq G^\infty$. Furthermore, we normalize the functions $w(z) = w(z; F)$ and $\tilde{w}(z) = w(z; \tilde{F})$ by the condition $w(0) = \tilde{w}(0)$. Then*

$$T_\alpha(\tilde{w}) \geq T_\alpha(w).$$

This theorem is a consequence of Littlewood's principle of subordination.[1] Under our assumptions $w(z)$ is subordinate to $\tilde{w}(z)$, i.e. there exists a function $\omega(z)$, regular for $|z|<1$ such that

$$|\omega(z)| \leq 1,$$

$$\omega(0) = 0$$

and

$$w(z) = \tilde{w}(\omega(z)). \tag{5.1}$$

Furthermore this function is schlicht. According to Schwarz's lemma we have

$$|\omega(z)| \leq |z| \text{ in } |z|<1. \tag{5.2}$$

If $z_\nu(a)$, $\nu = 1, 2, \ldots$, denote the zeros of $w(z) - a$, it follows from (5.1) and the fact that $\omega(z)$ is schlicht that

$$\tilde{z}_\nu(a) = \omega(z_\nu(a)), \quad \nu = 1, 2, \ldots,$$

are different a-values of $\tilde{w}(z)$. As furthermore by (5.2)

$$|\tilde{z}_\nu(a)| \leq |z_\nu(a)|,$$

we see that

$$\tilde{n}(r, a) \geq n(r, a)$$

with obvious notations. The theorem is thus proved.

The other general theorem is well-known, but since I have not seen it explicitly proved, I think it may be advisable to give a proof here.

Theorem 5.2: *Let F be a compact set of positive logarithmic capacity and normalize $w(z) = w(z; F)$ by the condition $w(0) = \infty$. If we define the functions of a set*

$$\nu_r(e) = \frac{1}{2\pi} m_1 E\{\Theta; w(r e^{i\Theta}) \in e\},$$

then

$$\lim_{r=1-0} \nu_r(e) = \mu_0(e)$$

exists, and μ_0 is the Robin distribution of the set F.

[1] Littlewood [2], page 163 ff.

If $z(w)$ is the inverse function of $w(z)$ and a_ν, $\nu = 0, 1, 2, .., a_0 = 0$, are its poles, then

$$G(w; \infty) = \sum_{\nu=0}^{\infty} \log \left| \frac{1 - z(w)\,\bar{a}_\nu}{a_\nu - z(w)} \right|$$

is the Green's function corresponding to F and singular at infinity.[1] If then

$$w(z) = \frac{c}{z} + c_0 + c_1 z + \ldots, \; |z| < \delta,$$

we conclude that

$$G(w; \infty) = \log |w| + \sum_{1}^{\infty} \log \frac{1}{|a_\nu|} - \log |c| + o(1), \; |w| \to \infty,$$

and we get according to nr. 6 with the same notations

$$V_0 = \sum_{1}^{\infty} \log \frac{1}{|a_\nu|} - \log |c|,$$

and for the classical definition of capacity the elegant formula

$$|c| \cdot \prod_{1}^{\infty} |a_\nu|.$$

If now a is a point in F, we have by Jensen's formula

$$\log |c| = \frac{1}{2\pi} \int_0^{2\pi} \log |w(r e^{i\Theta}) - a| \, d\Theta + \sum_{0 < |a_\nu| < r} \log \frac{r}{|a_\nu|} + \log r$$

or

$$\int \log \left| \frac{1}{w - a} \right| d\nu_r(w) = \sum \log \frac{r}{|a_\nu|} - \log |c| + \log r.$$

We can choose a sequence $\{r_n\}_1^\infty$, $\lim_{n=\infty} r_n = 1$, so that the corresponding set functions $\nu_{r_n} = \nu_n$ converge to a limit distribution ν, as $n \to \infty$. Since $\lim_{r=1-0} w(r e^{i\Theta})$ belongs to F whenever it exists and thus for almost all Θ — $w(z)$ belonging to T_0 — we have $\nu(F) = 1$

[1] See Nevanlinna [1], page 203.

and $\nu(CF) = 0$. The kernel $\log \left| \frac{1}{w-a} \right|$ is lower semi-continuous in every bounded domain and the difficulties arising from its singularity at infinity are easily eliminated. We conclude

$$\int_{\bar{F}} \log \left| \frac{1}{w-a} \right| d\nu(w) \leq \overline{\lim_{n=\infty}} \int \log \left| \frac{1}{w-a} \right| d\nu_n(w) = V_0$$

for all a belonging to F. According to the general maximum principle, this inequality holds for every a, and from the definition of V_0 and the uniqueness of the Robin distribution, we conclude that

$$\nu(e) \equiv \mu_0(e)$$

and also that the original sequence ν_r is convergent. The theorem is thus completely proved.

25. In this number we will use the fact that $w(z; F)$ is automorphic under a group Γ of linear transformations of the unit circle into itself to obtain certain properties of the functions $w(z; F)$ belonging to a class T_a. An immediate consequence of this fact is the following theorem.

Theorem 5.3: *If $w(z; F)$ belongs to T_a, then the series*

$$\sum_{1}^{\infty} (1 - r_\nu(a))^{1-a} \tag{5.3}$$

converges for every a, where $r_\nu(a)$, $\nu = 1, 2, \ldots$, denote the moduli of the a-values.

The series is convergent for almost all a. We may assume that it converges for $a = w(0)$ for, if it does not, we can make a linear transformation which does not affect the convergence or divergence of (5.3). We suppose that Γ consists of the transformations $S_\nu(z)$

$$\Gamma: \quad \begin{cases} S_0(z) \equiv z, \quad a_0 = 0, \\ S_\nu(z) = e^{ia_\nu} \dfrac{a_\nu - z}{1 - z\,\bar{a}_\nu}, \quad \nu = 1, 2, \ldots. \end{cases} \tag{5.4}$$

If now $w(z_0) = a$, we find

$$\sum_1^\infty (1-r_\nu(a))^{1-\alpha} < \sum_0^\infty (1-|S_\nu(z_0)|^2)^{1-\alpha} =$$

$$= (1-|z_0|^2)^{1-\alpha} \cdot \sum_0^\infty \frac{(1-|a_\nu|^2)^{1-\alpha}}{|1-z_0 a_\nu|^{2-2\alpha}} \leq A(z_0) \sum_0^\infty (1-|a_\nu|)^{1-\alpha} < \infty$$

according to the hypothesis, and our proposition is proved.

There is a converse of this theorem which will be needed later in the following form.

Theorem 5.4: *If $w(z)$ represents $|z|<1$ on the universal covering surface of a bounded domain D and if the series*

$$\sum_1^\infty (1-r_\nu(a))^{1-\beta}, \quad 0<\beta<\frac{1}{2},$$

converges for one point a in D, then $w(z)$ belongs to every class T_α with $\alpha<\beta$.

Let E be the set of a-values for which

$$\sum_1^\infty (1-r_\nu(a))^{1-\alpha} \leq 1, \quad \alpha<\beta,$$

and $H(r)$ the set of Θ-values where $w(re^{i\Theta})$ takes values not belonging to E. Then

$$\int_0^1 \frac{dr}{(1-r)^\alpha} \int_0^r r\,dr \int_0^{2\pi} |w'(re^{i\Theta})|^2\,d\Theta = I_1 + I_2 =$$

$$= \iint_E d\sigma_a \int_0^1 \frac{n(r,a)}{(1-r)^\alpha}\,dr + \int_0^1 \frac{dr}{(1-r)^\alpha} \int_0^r r\,dr \int_{H(r)} |w'(re^{i\Theta})|^2\,d\Theta.$$

The integral I_1 is finite by choice of the set E. In order to get an estimation of I_2, we introduce the group Γ under which $w(z)$ is automorphic. If Γ is given in (5.4), our assumptions yield in the same way as above that

$$\sum_0^\infty (1-|a_\nu|)^{1-\beta} < \infty,$$

and we find

$$m_1 H(r) \leq \int_{H(r)} \sum_{\nu=0}^{\infty} (1 - |S_\nu(r e^{i\Theta})|)^{1-\beta} \, d\Theta <$$

$$< (1-r^2)^{1-\beta} \sum_{\nu=0}^{\infty} (1-|a_\nu|^2)^{1-\beta} \int_0^{2\pi} \frac{d\Theta}{|1-\bar{a}_\nu r e^{i\Theta}|^{2-2\beta}} = O((1-r)^\beta), \quad r \to 1.$$

Observing that $|w'(z)| < \frac{A_1}{1-|z|}$, we obtain the following estimation of I_2:

$$I_2 \leq A_1^2 \int_0^1 \frac{dr}{(1-r)^\alpha} \int_0^r r \, dr \int_{H(r)} \frac{d\Theta}{(1-r)^2} \leq$$

$$\leq A_2 \int_0^1 \frac{dr}{(1-r)^\alpha} \int_0^r \frac{dr}{(1-r)^{2-\beta}} = A_3 \int_0^1 \frac{dr}{(1-r)^{1+\alpha-\beta}} < \infty,$$

where the A_i are certain constants, and the theorem is proved.

26. For our next theorem we need further information about the series $\sum_1^\infty (1-r_\nu(a))^{1-\gamma}$. We shall prove this

Lemma: *If* $w(z;F) \in T_\alpha$, $\frac{1}{2} < \alpha < 1$, *then there exists a function* $\varphi(r)$ *so that*

(i) $$n(r,a) \leq \varphi(r) \quad \textit{for all } a,$$

(ii) $$\int_0^1 \frac{\varphi(r)}{(1-r)^\beta} \, dr < \infty, \quad \beta = 2\alpha - 1,$$

where $n(r,a)$ *has its usual meaning.*

According to theorem 5.3, $\sum_1^\infty (1-r_\nu(a))^{1-\alpha}$ converges for every a, and this lemma will imply that $\sum_1^\infty (1-r_\nu(a))^{2(1-\alpha)}$ is uniformly

bounded. The restriction $\alpha \geq \frac{1}{2}$ is essential for the validity of any result of this kind with $\beta > 0$, as the instance

$$w(z) = e^{\frac{1+z}{1-z}}$$

shows. Se below, page 75.

In the proof we again use the fact that $w(z; F)$ is automorphic and suppose that the group Γ is given by (5.4). If $n(\eta)$ denotes the number of points a_ν in $|z| \leq 1 - \eta$, then according to theorem 5.3

$$\int_0^1 \frac{n(\eta)}{\eta^\alpha} d\eta < \infty.$$

Let a be any complex number and $z_\nu(a)$, $\nu = 0, 1, \ldots, |z_0| \leq |z_1| \leq \ldots$, the roots of the equation $w(z) - a = 0$. From the group properties of Γ it is easily seen that we can assume the sequence $\{S_\nu(z)\}$ to be numbered in such a manner that

$$z_\nu(a) = S_\nu(z_0), \quad \nu = 0, 1, 2, \ldots$$

Then

$$1 - |z_\nu|^2 = 1 - |S_\nu(z_0)|^2 = \frac{(1 - |a_\nu|^2)(1 - |z_0|^2)}{|1 - z_0 \bar{a}_\nu|^2} \leq 1 - |z_0|^2$$

for all ν. We observe that

$$|1 - z_0 \bar{a}_\nu|^2 > (1 - |z_0|)^2.$$

If r is a given number, $|z_0| \leq r < 1$, then the inequality

$$(1 - |z_0|^2) \frac{1 - |a_\nu|^2}{|1 - z \bar{a}_\nu|^2} \geq 1 - r^2$$

is possible only for a number of ν-values

$$\leq n\left[\frac{1}{2}(1 - |z_0|)(1 - r)\right] \leq n\left[\frac{1}{2}(1 - r)^2\right].$$

We get

$$\begin{cases} n(r, a) = 0 \quad \text{for} \quad r < |z_0|, \\ n(r, a) \leq n\left[\frac{1}{2}(1 - r)^2\right], \quad r \geq |z_0|. \end{cases}$$

Choosing

$$\varphi(r) = n\left[\frac{1}{2}(1-r)^2\right],$$

we find that the condition (i) is satisfied. Since

$$\int_0^1 \frac{\varphi(r)}{(1-r)^\beta}\,dr = \int_0^1 \frac{n\left[\frac{1}{2}\eta^2\right]}{\eta^\beta}\,d\eta = 2^{-\alpha}\int_0^{1/2} \frac{n(t)}{t^\alpha}\,dt,$$

our lemma is established.

We shall use this lemma to prove the following theorem.

Theorem 5.5: *If* $w(z; F) \in T_\alpha$, $\frac{1}{2} < \alpha < 1$, *then*

$$\text{Log-dim } (F) \geq \frac{1}{2(1-\alpha)}.$$

The proof is quite simple. We may suppose that $w(0; F) = \infty$, and conclude from the lemma and the remark of theorem 3.7 that

$$\frac{1}{2\pi}\int_0^{2\pi} \left\{ \overset{+}{\log} \left| \frac{1}{w(r\,e^{i\Theta}) - a} \right| \right\}^p d\Theta \leq M_p$$

for any $p < \frac{1}{2(1-\alpha)}$, where M_p is independent of $r < 1$ and a. Since the kernel is lower semi-continuous, we get, when $r \to 1$,

$$\int_F \left\{ \overset{+}{\log} \left| \frac{1}{w-a} \right| \right\}^p d\mu_0(w) \leq M_p, \tag{5.5}$$

where μ_0 is the Robin distribution. From theorem 1.4 we then conclude that

$$l_p(F) > 0,$$

and our proposition is proved.

From the inequality (5.5) we can also conclude that the Green's function for the complement of F is uniformly continuous outside a neighbourhood of the singularity, if $w(z; F) \in T_\alpha$, $\alpha > \frac{1}{2}$.

27. We shall now prove an auxiliary theorem which does not only give us information about the mapping functions in T_a but also proves theorem 2.6.

Theorem 5.6: *Let $w(z)$ be bounded and holomorphic in $|z|<1$ and belong to T_a, $0<a<1$. Suppose that the range D of $w(z)$ is bounded by the set S. If it is possible to write $S=S'+E$, where S' is a closed set such that $\lim_{r=1-0} w(re^{i\Theta})$ belongs to S' for almost all Θ, then*

$$l_{1-a}(E)=0.$$

We first prove two simple lemmas.

Lemma 1: *If $w(z)=\sum_0^\infty a_n z^n$ belongs to T_a and is bounded in $|z|<1$, then*

$$I(r)=\frac{1}{2\pi}\int_0^{2\pi}|w(e^{i\Theta})-w(re^{i\Theta})|^2\,d\Theta=o(\eta^a),\ \eta=1-r,$$

where $w(e^{i\Theta})=\lim_{r=1-0} w(re^{i\Theta})$ a.e.

Observing that $1-r^n\le n(1-r)$, $r<1$, $n\ge 1$, we get, if $N=\left[\frac{1}{\eta}\right]$,

$$I(r)=\sum_1^\infty |a_n|^2(1-r^n)^2\le\eta^2\sum_1^N n^2|a_n|^2+\sum_{N+1}^\infty |a_n|^2\le$$

$$\le\eta^a\cdot\sum_1^N |a_n|^2 n^a\cdot\left(\frac{n}{N}\right)^{2-a}+\eta^a\sum_{N+1}^\infty |a_n|^2 n^a=o(\eta^a),$$

since, according to our hypothesis,

$$\sum_1^\infty |a_n|^2 n^a<\infty.$$

Lemma 2: *If $w(z)$ is holomorphic, bounded and $\neq a$ in $|z|<1$, then*

$$|w(re^{i\Theta})-a|\ge e^{-\frac{A}{\eta}},\ \eta=1-r,$$

where A is a constant, independent of a.

By the Poisson formula we have for all ϱ, $r<\varrho<1$,

$$\log\frac{1}{|w(re^{i\Theta})-a|}=\frac{1}{2\pi}\int_0^{2\pi}\frac{\varrho^2-r^2}{\varrho^2+r^2-2r\varrho\cos(\Theta-\varphi)}\log\frac{1}{|w(\varrho e^{i\varphi})-a|}d\varphi,$$

whence

$$\log\frac{1}{|w(re^{i\Theta})-a|}\leq\frac{2}{1-r}\lim_{\varrho=1-0}\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{|w(\varrho e^{i\varphi})-a|}d\varphi\leq\frac{A}{\eta},$$

where, by Nevanlinna's fundamental theorem, A can be chosen independently of a.

Let D' be the domain bounded by S' and $g(w;a)$ be the Green's function for D', singular at $w=a$. If D_ε' is the set of points in D' with a distance to $S'>\varepsilon$, then there is a constant $K(\varepsilon)$ such that for all w and a in D_ε' it holds true that

$$g(w;a)\geq K(\varepsilon)>0.$$

Let now $\varepsilon>0$ be an arbitrary number and E_ε the subset of E belonging to D_ε'. If ε is so small that $w(0)$ belongs to $D_{2\varepsilon}'$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}g(w(re^{i\Theta});a)\,d\Theta=g(w(0);a)\geq K(2\varepsilon)=K.$$

for all a belonging to $E_{2\varepsilon}$, since by hypothesis $w(z)\neq a$ in $|z|<1$.

According to known properties of the Green's function

$$\lim_{w\to w^*}g(w;a)=0 \tag{5.6}$$

for all w^* in S' with the possible exception of a set of logarithmic capacity zero. The limit (5.6) exists uniformly for all a in $E_{2\varepsilon}$. By a theorem of Nevanlinna and Frostman [1] $\lim_{r=1-0} w(re^{i\Theta})$ can belong to the above-mentioned exceptional set only for a set of Θ-values of measure zero; from this we conclude that

$$\int_{w\in D'-D_\varepsilon'}g(w(re^{i\Theta});a)\,d\Theta\leq K$$

for $\eta=1-r\leq\eta_0$, where η_0 can be chosen independently of a in $E_{2\varepsilon}$.

[1] Nevanlinna [1] page 198, Frostman [1] page 97.

If now $O = O(r; \varepsilon)$ is the set of Θ-values for which $w(r e^{i\Theta})$ belongs to D_ε', we find from lemma 1 that

$$\varepsilon^2 m_1 O = o(\eta^\alpha)$$

and for fixed ε

$$m_1 O = \delta^2(\eta)\, \eta^\alpha, \quad \delta(\eta) \to 0, \ \eta \to 0. \tag{5.7}$$

If we use this and the evident inequality

$$g(w; a) \leq \log \frac{1}{|w - a|} + constant,$$

valid for all w and a in D', we see that there exists a number $\eta_1 > 0$ so that

$$\int_O \log \frac{1}{|w(r e^{i\Theta}) - a|}\, d\Theta \geq C > 0 \tag{5.8}$$

for $\eta \leq \eta_1$ and all $a \in E_{2\varepsilon}$, C being a certain constant. We can now apply theorem 1.3 with $\sigma = e^{-\frac{A}{\eta}}$, because of lemma 2, and find

$$\frac{1}{m_1 O} \int_O \log \frac{1}{|w(r e^{i\Theta}) - a|}\, d\Theta < \int_\sigma^1 \log \frac{1}{t}\, d \frac{1}{\delta(\eta)\left\{\log \frac{1}{t}\right\}^{1-\alpha}} + \frac{\left(\log \frac{1}{\sigma}\right)^\alpha}{\delta(\eta)} \tag{5.9}$$

for all a except for those a lying in a set $E(\eta)$ such that

$$l_{1-\alpha}(E(\eta)) \leq 36\, \delta(\eta).$$

If we evaluate the integral to the right in (5.9) and observe (5.7), we can conclude that there exists a constant A_1 so that

$$\int_O \log \frac{1}{|w(r e^{i\Theta}) - a|}\, d\Theta \leq A_1\, \delta(\eta)$$

except when $a \in E(\eta)$. Comparing this with (5.8), we find that for $\eta \leq \eta_2$, $E_{2\varepsilon}$ must be included in $E(\eta)$ whence

$$l_{1-\alpha}(E_{2\varepsilon}) \leq 36\ \delta(\eta), \ \eta \leq \eta_2,$$

and

$$l_{1-\alpha}(E_{2\varepsilon}) = 0.$$

Since ε was arbitrarily small and a countable union of null-sets is a null-set, we find

$$l_{1-\alpha}(E) = 0,$$

and the theorem is proved.

28. We use this auxiliary theorem to prove the following result.

Theorem 5.7: *Let $w(z; F)$ belong to T_α and suppose that $F = F' + R$, where F' is a closed set. Then*

(I) $$l_{1-\alpha}(R) = 0, \quad 0 < \alpha \leq \frac{1}{2}, \quad \text{if} \quad C(R) = 0,$$

(II) $$R \text{ is empty, if } \operatorname{Log-dim}(R) < \frac{1}{2(1-\alpha)}, \quad \frac{1}{2} < \alpha < 1.$$

Theorem (E) of the introduction showed that when $\alpha = 0$, the metrical magnitude of the set F, i.e. its logarithmic capacity, completely determines whether or not $w(z; F)$ belongs to T_α. The preceding theorem brings out the remarkable fact that when $\alpha > 0$, this is no longer so, and now the small isolated subsets R of F play an important role. Furthermore, if $\alpha > \frac{1}{2}$, no such small subsets can exist. We shall see later on that $\alpha = \frac{1}{2}$ is the critical point.

To prove (I) we argue as follows:

Suppose that we have a division $F = F' + R$ such that $C(R) = 0$ and $l_{1-\alpha}(R) > 0$. Let a_0 be an arbitrary point in F'. We make a rotation of the sphere so that a_0 falls on the point at infinity. The function $w(z; F)$ turns into a function $\tilde{w}(z)$, F' into $\tilde{F}'$ and R into $\tilde{R}$, where the new sets are of the same kind as the original ones. Now there exists a number ϱ such that the subset E of $\tilde{R}$ which lies in $|w| < \varrho$, has positive $l_{1-\alpha}$-measure. If $w(z)$ maps $|z| < 1$ onto the universal covering surface of the domain bounded by $|w| = \varrho$, $\tilde{F}'$ and E, we conclude from theorem 5.1 that $w(z)$ belongs to T_α, and it satisfies all the conditions in theorem 5.6 by the theorem of

Nevanlinna and Frostman cited on page 68. It follows that $l_{1-\alpha}(E) = 0$, which is a contradiction and proves the assertion.

In the case (II) we assume again that such a division of F exists. We first prove that

$$\lim_{r=1-0} w(r e^{i\Theta}; F)$$

belongs to F' for almost all Θ. Let H be the Borel set where this limit belongs to R and suppose that $m_1 H > 0$. If $w(z; F)$ is normalized by the condition $w(0; F) = \infty$, then by theorem 5.2 $\mu_0(R) > 0$ where μ_0 is the Robin distribution for the set F. We can thus choose the constant K so that

$$\mu_1(e) = K \cdot \mu_0(e R)$$

is a distribution of the unit mass on R. But observing (5.5), we conclude that

$$\int_R \left\{ \overset{+}{\log} \frac{1}{|w-a|} \right\}^p d\mu_1(w) \leq M_p', \quad p < \frac{1}{2(1-\alpha)},$$

and from theorem 1.4 it follows that $l_p(R) > 0$ for all $p < \frac{1}{2(1-\alpha)}$, and this contradicts our assumption.

We now suppose that R contains the point a. In the same way as above, we construct the bounded function $w(z)$. Let a' be the point in E corresponding to a. Then by (5.8) it holds true that, $1 < p < \frac{1}{2(1-\alpha)}$,

$$0 < C \leq \int_O \overset{+}{\log} \frac{1}{|w(r e^{i\Theta}) - a'|} d\Theta \leq$$

$$\leq \left\{ \int_O \left\{ \overset{+}{\log} \frac{1}{|w(r e^{i\Theta}) - a'|} \right\}^p d\Theta \right\}^{\frac{1}{p}} \left\{ m_1 O \right\}^{1-\frac{1}{p}}$$

which leads to a contradiction if we let $r \to 1$, and so our proposition follows.

29. These sets R with $C(R) = 0$ satisfy, when $w(z; F) \in T_a$, another rather strong condition which admits a more intuitive interpretation as a restriction of the ramification of the Riemann surface.

We shall introduce a function $\sigma_E(t)$ which gives us information about the extent to which the closed set E is "spread out" in the plane. Let γ be a connected curve having length $|\gamma|$ such that every point in E has a distance $\leq t$ from γ. We define

$$\sigma_E(t) = \inf_{\gamma} |\gamma|$$

for all such curves γ. If, thus, E consists of the two points a and b, then

$$\lim_{t=+0} \sigma_E(t) = |a - b|,$$

and if E has positive two-dimensional measure, then

$$\lim_{t=+0} t \cdot \sigma_E(t) \geq \frac{1}{2} m_2 E.$$

We now have the following theorem.

Theorem 5.8: *Suppose that* $w(z; F) \in T_a$, $0 < \alpha \leq \frac{1}{2}$, *and that* $F = F' + R$, *where* F' *is closed and* $C(R) = 0$. *Then for any closed subset* E *of* R

$$\int_0 \sigma_E^2(t) \frac{dt.}{t\left(\log \frac{1}{t}\right)^{\frac{2}{\alpha}}} < \infty. \tag{5.10}$$

In the same way as above we make a rotation of the sphere and introduce the bounded function $w(z)$. The set E becomes a bounded set $\tilde{E}$, and we may investigate this set instead.

Let γ_r be the image under $w(z)$ of $|z| = r$. If $L(r)$ is its length, we obtain by Schwarz's inequality

$$L(r)^2 = \left(\int_0^{2\pi} |w'(r e^{i\Theta})| \, r \, d\Theta \right)^2 \leq 2\pi r \int_0^{2\pi} |w'(r e^{i\Theta})|^2 \, r \, d\Theta.$$

Since $w(z) \in T_\alpha$, it follows that

$$\int^1 L(r)^2 (1-r)^{1-\alpha}\, dr < \infty. \tag{5.11}$$

For every point a in $\tilde{E}$, we have

$$\inf_{w \in \gamma_r} |a - w| \leq e^{-(1-r)^{-\alpha}}, \quad r > r_0.$$

This is an immediate consequence of the inequality (5.8), which is valid for all a in $\tilde{E}$, since

$$-m_1 O \cdot \log(\inf |w - a|) \geq C > 0$$

and

$$m_1 O = o((1-r)^\alpha).$$

We conclude that the curve γ_r satisfies the conditions by definition of $\sigma_{\tilde{E}}(t)$ for $t = e^{-(1-r)^{-\alpha}}$ and thus

$$L(r) \geq \sigma_{\tilde{E}}\left(e^{-(1-r)^{-\alpha}}\right), \quad r > r_0.$$

If this inequality is inserted in (5.11), our proposition follows for $\tilde{E}$ and hence for E.

We construct the following countable closed set E_0 for which it is easily verified that (5.10) does not hold for any $\alpha > 0$.

$$E_0: \quad \begin{cases} a_n^{(k)} = \dfrac{1}{\log n}\, e^{\frac{2\pi i k}{n}}, & k = 0, 1, \ldots, n-1, \\ & n = 2, 3, \ldots \\ 0 \end{cases}$$

If now, for instance, F consists of the circle $|z - 2| \leq 1$ and E_0, $w(z; F)$ does not belong to any class T_α with $\alpha > 0$, while $m_2 F$ is positive.

30. We have seen that if $w(z; F) \in T_\alpha$, $\alpha > \frac{1}{2}$, then F does not contain any isolated set R of vanishing logarithmic capacity. In the limit case $\alpha = \frac{1}{2}$ we can prove the following simple theorem.

Theorem 5.9: *If* $w(z; F) \in T_{\frac{1}{2}}$, *then* F *is perfect.*

We suppose on the contrary that F contains an isolated point a. When w describes a small circle around a, the inverse function $z(w)$ of $w(z; F)$ is subjected to a parabolic substitution.[1] The group under which $w(z; F)$ is automorphic must thus contain a parabolic substitution $S(z)$. We shall prove that this is impossible if $w(z; F) \in T_{\frac{1}{2}}$.

We can write the substitution $S(z)$ in the form

$$\frac{1}{\zeta - c} = \frac{1}{z - c} + i\lambda\bar{c}, \tag{5.12}$$

where $|c| = 1$ and λ is a real number different from zero. If we now set

$$z_n = S^n(0),$$

where S^n is S applied n times in succession, then (5.12) gives

$$\frac{1}{z_n - c} = \frac{1}{z_{n-1} - c} + i\lambda\bar{c}, \quad z_0 = 0,$$

and after addition

$$\frac{1}{z_n - c} = -\frac{1}{c} + n\, i\lambda\bar{c} = \frac{i\, n\lambda - 1}{c}$$

and

$$1 - |z_n|^2 = \frac{1}{n^2\lambda^2 + 1}.$$

We find that $\sum_1^\infty (1 - |z_n|)^{\frac{1}{2}}$ diverges and *a fortiori* that, putting $w(0; F) = a_0$,

$$\sum_1^\infty (1 - r_\nu(a_0))^{\frac{1}{2}}$$

diverges, which by theorem 5.3 is not possible if $w(z; F)$ belongs to $T_{\frac{1}{2}}$.

We shall show that $\alpha = \frac{1}{2}$ really separates the two cases. The function

[1] See e.g. Kaila [1], page 53.

$$w(z) = e^{\frac{z+1}{z-1}}$$

is easily seen to map the unit circle onto the universal covering surface of the domain bounded by the origin and the circle $|w| = 1$. Setting

$$e^{\frac{z+1}{z-1}} = \sum_0^\infty a_n z^n,$$

we find, e.g. by use of the saddle-point method, that

$$|a_n| = O(n^{-\frac{3}{4}}),$$

whence

$$\sum_1^\infty |a_n|^2 n^\alpha < \infty,$$

if $\alpha < \frac{1}{2}$. However, we do not insist on the details.

31. We shall finally study some examples, which will give us information about certain of the properties of the set F that affect the regularity of $w(z; F)$ as measured by the quantities $T_\alpha(w)$. These examples also show that the connectivity of the corresponding domain is not decisive.

Theorem 5.10: *There exist for every $\alpha < 1$ domains of infinite connectivity such that the corresponding mapping functions belong to T_α.*

We start from a non-increasing sequence of positive real numbers $\{a_\nu\}_1^\infty$ such that $\sum_1^\infty a_\nu < \infty$. On the real axis we define the closed intervals I_ν such that a_ν is the length of I_ν and I_ν is to the right of $I_{\nu-1}$ and divided from it by an interval L_ν of length $2l_\nu$, $l_\nu < a_\nu$. Then the intervals tend to a certain point P. We choose

$$F = P + \sum_1^\infty I_\nu$$

and shall prove that the numbers l_ν can be chosen so that $w(z; F)$ belongs to a given class T_α.

If $z(w)$ is the inverse function of $w(z; F)$, we study the function

$$u(w) = -\log|z(w)|,$$

which is harmonic on the covering surface except for a logarithmic singularity. We continue $u(w)$ along all paths in the w-plane, starting at the singularity, and call two paths equivalent if they cross the same intervals L_ν in the same order and direction. Let $P(t)$ be the number of non-equivalent paths on which $u(w) \geq t > 0$. It is easily seen that

$$\frac{1}{\pi} \iint\limits_{-\log|z| \geq t} \frac{|w'(z)|^2}{(1+|w(z)|^2)^2}\, d\sigma_z \leq P(t). \tag{5.13}$$

We have to estimate how rapidly $u(w)$ decreases as w moves from one interval L_ν to another. This is done by the following lemma.

Lemma 1: *If $\omega(z)$ is harmonic outside the interval I, $(-1, 1)$, on the real axis and $\omega(z) = 1$ on $(-\delta, \delta)$, $\delta < 1$, $\omega(z) = 0$ on the rest of I, then for all real x outside I*

$$\omega(x) < \delta.$$

The maximum is attained at infinity, and it is easily shown, e.g. by a conformal mapping onto the unit circle, that

$$\omega(\infty) = \frac{2}{\pi} \arcsin \delta < \delta.$$

We now put

$$\delta_\nu = \frac{l_\nu}{a_\nu}.$$

We continue $u(w)$ along a curve γ and suppose that $u(w) \leq \lambda$ when γ crosses L_μ. Since $u(w) = 0$ on the intervals I_ν, we conclude from the maximum principle for harmonic functions and the last lemma that

$$u(w) < \delta_\mu \cdot \lambda,$$

when we have continued $u(w)$ to another interval $L_{\mu'}$. We thus find that there is an $\eta > 0$ such that $\eta P(t)$ is not greater than the number of solutions of the inequality

$$\prod \delta'_\nu \geq t$$

or, in other words,

(5.14) $$\sum \log \frac{1}{\delta'_\nu} \leq \log \frac{1}{t},$$

where $\delta'_{2\nu-1} = \delta'_{2\nu} = \delta_\nu$, and where two solutions are identical only if the indices ν follow in the same order. We use

Lemma 2: *Suppose that* $A_\nu \geq 1$, $\nu = 1, 2, \ldots$, *and that* $N_p(T)$ *is the number of solutions of*

$$\sum A_\nu \leq T, \quad \nu \leq p.$$

If

$$N_p(T) \leq K \cdot e^{c_p \cdot T},$$

then

$$N_{p+1}(T) \leq K\, e^{c_{p+1}\, T}, \text{ where}$$

$$c_{p+1} = c_p + e^{-c_p A_{p+1}}.$$

For putting $A = A_{p+1}$ we obtain

$$N_{p+1}(T) \leq N_p(T) + N_p(T - A)\binom{[T]}{1} + N_p(T - 2A)\binom{[T]}{2} + \ldots \leq$$

$$\leq K\, e^{c_p T}\left(1 + \binom{[T]}{1} e^{-c_p A} + \binom{[T]}{2} e^{-2c_p A} + \ldots\right) \leq$$

$$\leq K\, e^{c_p T}(1 + e^{-c_p A})^T \leq K\, e^{T \cdot c_{p+1}}.$$

We observe that these estimations are rather rough.

If now $\alpha < 1$ is a given number and $1 - \alpha > 2\gamma > 0$, we choose the numbers δ_ν so that

(5.15) $$2 \sum_1^\infty \delta_\nu^\gamma < \gamma.$$

Let $Q_n(t)$ be the number of solutions of (5.14) when we consider only the first n numbers δ'_ν. Then

$$Q_1(t) \leq K \cdot t^{-\gamma}$$

for some constant K. If

$$Q_p(t) \leq K \cdot t^{-\gamma_p}, \quad \gamma_p \geq \gamma,$$

it follows from lemma 2 that

$$Q_{p+1}(t) \leq K \cdot t^{-\gamma_p - \delta'_p + 1} = K \cdot t^{-\gamma_{p+1}}.$$

From the assumption (5.15) it then follows that $\gamma_p < 2\gamma$ for all p and thus

$$\eta P(t) \leq K \cdot t^{-2\gamma}.$$

According to (5.13), the theorem is thus proved.

We shall also prove the following analogous theorem.

Theorem 5.11: *For every* $\alpha < \frac{1}{2}$, *there exist closed sets* F, *containing an infinity of isolated points, such that* $w(z; F) \in T_\alpha$.

In the same way as above, we can show that there exist two disjoint circles C_1 and C_2 and a point P outside them, such that the corresponding mapping function $w_1(z)$ belongs to T_α, where α is a given number less than $\frac{1}{2}$. We have only to choose the circles so close to each other that $u(w)$ decreases with sufficient rapidity when w passes between them. The details may be left to the reader.

We next consider the function $w_2(\zeta) = w(\zeta; F_2)$ where $F_2 = C_1 + C_2$. By the inverse function of w_2, $\zeta(w)$, P is mapped on an infinite sequence of points P_ν in $|\zeta| < 1$. Let D be the domain bounded by these points and the circle $|\zeta| = 1$, and suppose that $\varphi(z)$ maps $|z| < 1$ onto the universal covering surface of D. If $\varphi(z)$ is normalized suitably, then

$$w_1(z) = w_2(\varphi(z)).$$

When $\varphi(z) = a$, then $w_1(z) = w_2(a)$. If thus $w_1(z) \in T_\alpha$ and $r_\nu(a)$, $\nu = 1, 2, \ldots$, are the moduli of the a-values of $\varphi(z)$, we conclude from theorem 5.3 that

$$\sum_1^\infty (1 - r_\nu(a))^{1-\alpha} < \infty.$$

From theorem 5.4 it then follows that $\varphi(z) \in T_\beta$ for $\beta < \alpha$, and since α can be chosen arbitrarily close to $\frac{1}{2}$, the theorem is proved.

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